On the compactness of nonmonotonic logics.

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Abstract

A weak concept of compactness for nonmonotonic logics is proposed, which is suitable for several nonmonotonic logics, for which MAKINSONs smallest cumulative extension as well as FREUND/LEHMANNs canonical extension fail.

1. Introduction and Preliminaries

While for monotonic logics TARSKIs compactness axiom

$$Cn(X) := \bigcup_{\substack{Y \subseteq X \text{ and} \\ Y \text{ is finite}}} Cn(Y)$$

provides a canonical link between finitary and infinitary monotonic consequence operations, it seems to be much more difficult to find a condition suitable for nonmonotonic logics. MAKINSON, FREUND and LEHMANN [FLM90], [Mak93], [Fr93], [FL94] have proposed several concepts of compactness, i.e. , operators extending a finitary inference operation to an operation defined for arbitrary sets of wffs. Their construction are taylored to preserve certain properties, i.e., if a finitary operation F satisfies (the finitary version) of a condition such as cumulativity, then the extension of F should satisfy this condition too. Unfortunately, these extensions have an important drawback: it turns out, that very simple nonmonotonic logics like minimal reasoning in propositional logic (MRPL for short) are not compact with respect to these extensions. The first extension which is suitable for MRPL is Δ_4 introduced by HERRE/DIETRICH [DH94] and further investigated by HERRE [He95]. Since Δ_4 is not applicable to slight generalizations to MRPL, such as POOLE-systems without constraints and minimal reasoning in first order logic (circumscription), alternative extensions have to be investigated. We use a topological approach to define such an extension operator. The plan of the paper is to formulate a couple of principles, which should be satisfied by any extension, and evaluate the known extensions on the base of these principles.

Some remarks to the notions. With $\mathcal{L}$ we denote a logical language, i.e., in the most general setting, a countable set. Operations $C : 2^{\mathcal{L}} \to 2^{\mathcal{L}}$ satisfying the condition $\forall X \subseteq \mathcal{L} : X \subseteq C(X)$, are called inference operations. Two inference operations C_1, C_2 can be compared as follows: $C_1 \leq C_2$ iff $\forall X \subseteq \mathcal{L} : C_1(X) \subseteq C_2(X)$. $X \subseteq_f Y$ is standing for "X is a finite subset of Y"; $2_f^{\mathcal{L}}$ denotes the class of finite subsets of X. Given an operation $C : 2^{\mathcal{L}} \to 2^{\mathcal{L}}$, $C_f : 2_f^{\mathcal{L}} \to 2^{\mathcal{L}}$ denotes the finitary restriction of C , i.e., $C_f := C_{|2_f^{\mathcal{L}}}$.

Definition 1.1 (DH94). *Let $\mathcal{L}$ be a logical language. A structure $(\mathcal{L}, Cn, C)$ is said to be an* inferential frame *iff the following conditions are satisfied:*

- *$\mathcal{L}$ is an enumberable set called logical language.*
- *$Cn : 2^{\mathcal{L}} \to 2^{\mathcal{L}}$ is a deductive operation, i.e., it fulfills the* Tarski*-axioms inclusion, idempotence, monotony and compactness.*
- *$C : 2^{\mathcal{L}} \to 2^{\mathcal{L}}$ is an operation such that* supradeductivity *$Cn \leq C$ is satisfied.*

If C is replaced by a finitary operation $F : 2_f^{\mathcal{L}} \to 2^{\mathcal{L}}$ and supradeductivity is modified to $\forall X \subseteq_f \mathcal{L} : Cn(X) \subseteq C(X)$, then $(\mathcal{L}, Cn, C)$ is said to be a finitary inferential frame. *A (finitary) inferential frame is called cumulative (strongly cumulative) iff the operation C (F, respectively) satisfies (the finitary version of) cumulativity (strong cumulativity).*
The monotonic part of a (finitary) inferential frame $(\mathcal{L}, Cn)$ is said to be its base.

For monotonic logics, there are several presentations of the compactness axiom. From now, we will regard the condition

$A \in C(X)$ implies $A \in C(Y)$ for some finite subset of X.

as compactness.

Definition 1.2 (DH94). *Let **F** be a class of finitary inferential frames and **I** a class of inferential frames. An operator $\Delta : \mathbf{F} \to \mathbf{I}$, mapping finitary inferential frames to inferential frames $\Delta(\mathcal{L}, Cn, F) := (\mathcal{L}, Cn, C)$, is said to be an extension on **F** iff for any inferential frame $(\mathcal{L}, Cn, F) \in \mathbf{F}$ the following two conditions are satisfied:*

- *The bases of $(\mathcal{L}, Cn, F)$ and $\Delta(\mathcal{L}, Cn, F)$ coincide.*
- *The finitary restriction of C equals F: $C_f = F$.*

*Let $(\mathcal{L}, Cn, C) \in \mathbf{I}$ be an inferential frame and Δ an extension on **I**. $(\mathcal{L}, Cn, C)$ is said to be* completely Δ-compact *iff $(\mathcal{L}, Cn, C) = \Delta(\mathcal{L}, Cn, C_f)$.*

Since the base of an inferential frame is not changed by an extension, an extension may be also seen as an operator mapping finitary inference operations to infinitary inference operations, but depending on the additional parameter Cn. For this reason, we will use the notion $\Delta^{Cn} F$.

Example 1.3. *Deductive operations* $Cn : 2^{\mathcal{L}} \to 2^{\mathcal{L}}$, *may be considered as inferential frames*[1] $(\mathcal{L}, id, Cn)$, TARSKIs *compactness axiom provides the extension* Δ_{mon}

$$\Delta_{mon}^{id}(F) := \bigcup_{Y \subseteq_f X} F(Y).$$

There is no doubt about Δ_{mon} being indeed the canonical extension for monotonic logics, and should be preserved by any extension defined on a more comprehensive class of inferential frames. Therefore we should require the following principle for any extension :

Conservative extension In case of operations satisfying monotony, idempotence and inclusion, complete Δ-compactness and compactness should be equivalent conditions, i.e., $\Delta_{mon}^{id} Cn_f = \Delta^{id} Cn_f$.

2. Strong Principles: the smallest cumulative extension and the canonical extension

Δ_{mon} is obviously not suitable for nonmonotonic logics (inferential frames $\mathcal{IF} := (\mathcal{L}, Cn, C)$): even the existence of two finite sets $X \subseteq Y$, such that C is nonmonotonic on these sets, i.e., $A \in C(X)$ and $A \notin C(Y)$ for some wff A, is sufficient, that complete Δ_{mon}-compactness fails for $\mathcal{IF}$. The first alternative extensions have been proposed by MAKINSON [Mak93], FREUND [Fr93] and FREUND, LEHMANN [FL94]. We recall their definitions.

Definition 2.1 (Mak93,FL94). *Let* $(\mathcal{L}, Cn, F)$ *be a finitary inferential frame. The extensions* Δ_i^{Cn}, $i \leq 3$, *are defined by*

- $\Delta_1^{Cn} F(X) := \begin{cases} F(Y) & \text{wenn} \quad \exists Y \subseteq_f Cn(X) : Cn(X) \subseteq F(Y) \\ Cn(X) & \text{sonst} \end{cases}$.
- $\Delta_2^{Cn} F(X) := \{A \mid \exists Y \subseteq_f X \forall Z \subseteq_f Cn(X) : A \in F(Y \cup Z)\}$.
- $C_0(X) := \Delta_2^{Cn} F(X); C_{i+1}(X) := \{A \mid \exists Y \subseteq_f X \forall Z \subseteq C_i(X) : A \in C_i(Y \cup Z)\}$.
- $\Delta_3^{Cn} F(X) := \bigcap_{i \in \omega} C_i(X)$.

The following lemma follows immediately from the last definition.

Lemma 2.2. *Complete* Δ_1-, Δ_2- *and* Δ_3-*compactness, respectively, of a cumulative inferential frame* $(\mathcal{L}, Cn, C)$ *imply compactness of* C.

$\Delta_3^{Cn} F$ is also said to be the *canonical extension of* F [FL94], while $\Delta_1^{Cn} F$ is called the *smallest cumulative extension of* F[Mak93]. The last name illustrates

[1] id indeed satisfies TARSKIs axioms inclusion, idempotence, monotony and compactness.

the importance of the parameter Cn: indeed, it can be proved[2], that $\Delta_1^{Cn}F$ is the smallest cumulative extension of F, such that the additional constraint of supradeductivity with respect to Cn is satisfied. But different bases $(\mathcal{L}, Cn_1)$ $(\mathcal{L}, Cn_2)$ may produce different extensions $\Delta_1^{Cn_1}F$ and $\Delta_1^{Cn_2}F$ of F. However, it can be easily verified, that $Cn_1 \leq Cn_2$ implies $\Delta_1^{Cn_1}F \leq \Delta_1^{Cn_2}F$. Since the minimal possible base of F is the identity id defined by $id(X) := X$ for any set of wffs X, the smallest cumulative extension of F is given by:

$$\Delta_1^{id}F := \begin{cases} F(Y) & \text{if} \quad \exists Y \subseteq_f X : X \subseteq F(Y) \\ X & \text{otherwise} \end{cases}$$

FREUND and LEHMANN [FL94] have provided an example which shows that Δ_1 is not conservative. The smallest cumulative extension as well as the canonical extension have a couple of good properties such as preserving[3] cumulativity and other conditions. But there is an important drawback: $\Delta_1^{id} - \Delta_3^{id}$ do not provide compactness of one of the most natural nonmonotonic logic, which can be reconstructed in nearly all standard systems.

The inferential frame MRPL:= $(\mathcal{L}_{CPL}, Cn_{CPL}, C_{\min})$ is defined as follows: $\mathcal{L}_{CPL}$ is the language and Cn_{CPL} the deductive operation of classical propositional logic CPL, models of CPL are sets of atomic formulas. These models may be ordered by set inclusion, $Min(X)$ is the set of minimal models of a set X. Finally, $C_{\min}$ is defined by $C_{\min}(X) := ThMin(X)$.

Example 2.3. *MRPL is not weakly compact. The first counterexample has been provided by* PAPALASKARI *and* WEINSTEIN *[PW90], the following simple example presented here is taken from [DH94]. Put*

- $\mathcal{L}_{atom} := \{p_0, p_1, ..\}$.
- $X := \left\{ \bigwedge_{i=1}^{k} p_i \wedge (p_{i+1} \vee p_0) | k \in \omega \right\}$.
- $A := \neg(p_0 \leftrightarrow p_1)$.

X has the following two models:

$$v_1(p_i) := \begin{cases} 1 & \text{if} \quad i > 0 \\ 0 & \text{if} \quad i = 0 \end{cases} ; v_2(p_i) := 1 \text{ for every } i.$$

This yields $v_1 < v_2$ and thereby $A \in C_{\min}(X)$. Let X_f be a finite subset of X, and put $N := \max\{i | B_i \in X_f\}$. Then v_0, defined by

$$v_0(p_i) := \begin{cases} 1 & \text{if} \quad 0 \leq i \leq N \\ 0 & \text{if} \quad i > N \end{cases} ,$$

is a minimal model of X_f and it follows $A \notin C_{\min}(X_f)$.

[2] For a proof the reader is referred to [FL94].

[3] Some proofs require a certain expressiveness of the logical language, for details the reader is referred to [FL94].

3. Weak Principles

The last example leads to the following claim.

Principle of Weakness (PW) Complete Δ-compactness of an inferential frame should not imply its compactness.

The first weak extension is Δ_4 proposed in [DH94] and further investigated by HERRE [He95].

$$\Delta_4^{Cn} F(X) := \{A | \forall Y \subseteq_f Cn(X) \exists Z \subseteq_f Cn(X) : A \in F(Y \cup Z)\}$$

Δ_4 indeed satisfies the principle of weakness, as shown by the next theorem:

Proposition 3.1 (DH94,He95). *MRPL is completely Δ_4-compact.*

Sketch of Proof. The proof uses localization of models to finite sets of variables. Let $\mathcal{L}_{atom}$ be the set of atomic formulas and let $v \in 2^{\mathcal{L}_{atom}}$ be any model of classical propositional logic, i.e., a truth-false-valuation. The restriction of a elementary class of models $Mod(X)$ to a set of atoms $V \subseteq \mathcal{L}_{atom}$ is defined by

$$Mod_V(X) := \left\{ v \in 2^{\mathcal{L}_{atom}} | \exists v' \in Mod(X) : v \cap V = v' \cap V \right\}$$

If V is a finite, then there exits a formula H_V^X, such that $Mod_V(X) = Mod\left(\left\{H_V^X\right\}\right)$. Let $p_1, p_2, ..$ an enumeration of $\mathcal{L}_{atom}$ and $V_n := \{p_1, .., p_n\}$. The core of the proof is to show, that $A \in C_{\min}(X)$ iff $A \in C_{\min}\left(\left\{H_{V_n}^X\right\}\right)$ for every finite V_n containing the atoms occurring in A. $\square$

FREUND and LEHMANN have shown, that POOLE-systems[4] with a finite sets of defaults are completely Δ_2–compact. At least for POOLE-system based on CPL and certain many-valued logics, proposition can be generalized for certain infinite systems, in particular, if the set of constraints is empty and the set of defaults consists of literals only [DH94]. This is not longer true if we turn to systems with complex formulas as defaults.

Example 3.2. *There are* POOLE-*systems* $(\mathcal{L}_{CPL}, Cn_{CPL}, C_{(D,\emptyset)})$ *without constraints, which are not weakly compact. Let* X *and* A *be as defined in example 2.3 and put:*

- $\delta_i := \neg p_i \wedge (p_0 \vee p_i) \wedge (p_1 \vee p_i)$ for every $i \in \omega$.
- $D := \{\neg p_0\} \cup \{\delta_i\}_{i \geq 1}$.

The models of X are v_1 and v_2, as defined in example 2.3. The sets $X \cup \{\delta_i\}$ are inconsistent and hence we have $A \in Cn_{CPL}(X \cup \{\neg p_0\}) = C_{(D,\emptyset)}(X)$. Let X_f be a finite subset of $Cn_{CPL}(X)$. From $Mod(X) \subseteq Mod(X_f)$ follows the

[4]For the definition of POOLE-systems the reader is referred to [Po88] or [FL94].

existence of a model $m \in Mod(X_f)$, such that $m \models p_0$ and $m \models p_1$. Let p_N be an arbitrary variable not occurring in X_f. Then we may assume $m \models \neg p_N$. Hence $X_f \cup \{\delta_N\}$ has a model and is consistent. Let $d \subseteq D$ be a maximal set of defaults, such that $Cn_{CPL}(X_f \cup d) \neq \mathcal{L}_{CPL}$ and $\delta_N \in d$. Then we have $m \models p_0 \leftrightarrow p_1$ for every model $m \in Mod(X_f \cup d)$ and hence $A \notin Cn_{CPL}(X_f \cup d)$ and $A \notin C_{(D,\emptyset)}(X_f)$. Thereby $(\mathcal{L}_{CPL}, Cn_{CPL}, C_{(D,\emptyset)})$ is not weakly compact.

There are other examples showing that important nonmonotonic logics are not weakly compact, e.g. minimal reasoning in first order logic [DH94], [He95] and POOLE-systems with constraints and with a set of defaults consisting of literals only [DH94]. Hence an extension operator should satisfy the following principle:

Principle of Ultraweakness (PUW) Complete Δ-compactness of an inferential frame should not imply its weak compactness.

It follows immediately, that the extensions $\Delta_1 - \Delta_4$ are not ultra weak.

4. Pseudometric spaces

Example 4.1. *Let $\mathcal{L}_{CPL}$ be the language of classical propositional logic and $\mathcal{M}_0 \subseteq 2^{\mathcal{L}_{atom}}$ the class of finite and cofinite sets of atomic formulas of $\mathcal{L}$. The relation $\models \subseteq \mathcal{M}_0 \times \mathcal{L}_{CPL}$ is defined as follows:*

- *$m \models p$ iff $p \in m$.*
- *$m \models A \vee B$ iff $m \models A$ or $m \models B$.*
- *$m \models A \wedge B$ iff $m \models A$ and $m \models B$.*
- *$m \models \neg A$ iff $m \not\models A$.*

Then $\mathcal{S} := (\mathcal{L}_{CPL}, \mathcal{M}_0, \subset, \models)$ is a finitary model structure *in the sense as follows: the relation $\subset$ is irreflexive and antisymmetric, and $\mathcal{S}$ is stoppered for finite sets X, i.e., any element of $\hat{X} := \{m \in \mathcal{M}_0 | \forall A \in X : m \models A\}$ is either minimal or an extension of a minimal element of $\hat{X}$. We verify the last claim. Let $V \subseteq \mathcal{L}_{atom}$ be the finite set of atomic formulas occurring in X. Whether a set $m \in \mathcal{M}_0$ is in $\hat{X}$ depends only on the occurrence of elements of V in m. Thereby $m \in \hat{X}$ implies $m \cap V \in \hat{X}$. By finiteness of V we conclude the existence of minimal elements in $\hat{X}$.*

Now consider the infinite set of wffs $X := \{p_{2i} \vee p_{2i-1}\}_{i \geq 1}$. Then $m \in \hat{X}$ iff for every i holds $\{p_{2i}, p_{2i-1}\} \cap m \neq \emptyset$. Thereby the elements of the infinite chain `chain`

$$\texttt{chain} : \mathcal{L}_{atom} \supset \mathcal{L}_{atom} \backslash \{p_2\} \supset \mathcal{L}_{atom} \backslash \{p_2, p_4\} \supset .. \supset \mathcal{L}_{atom} \backslash \{p_{2i}\}_{i \leq n} \supset ..$$

are $\hat{X}$, since they are cofinite. Moreover, $\hat{X}$ has no minimal models, since minimality of m implies that m selects precisely one elements from each pair

$\{p_{2i}, p_{2i-1}\}$ *and is neither finite nor infinite. It follows, that* $\mathcal{S}$ *is not a MAK-model structure. It can be easily verified that the finitary inference operation generated by* $\mathcal{S}$ *is (finitary) MRPL. The question arises, whether* $\mathcal{S}$ *completed by gaps like the infimum of* `chain` *generates full MRPL.*

The idea of the new extension introduced in this paper is as follows: we consider the space $\mathcal{S} := \{C(X) | X \subseteq_f \mathcal{L}\}$ and add all gaps which are approached by sequences like `chain` in the example above, the result of this completion procedure is denoted by $\mathcal{S}^*$. These gaps are considered as representations of C-closed sets of wffs $C(Y)$, where Y may be an infinite set. In the second step we have to define, how an element from $\mathcal{S}^*$ is selected for any infinite set of wffs. In case of $\mathcal{S}$, $F(X)$ is determined with respect to certain relations $\prec \subseteq \mathcal{S} \times \mathcal{S}$ and $\models \subseteq \mathcal{S} \times \mathcal{L}$, this is precisely MAKINSONs semantics for cumulative inference operations [Mak89]. By adapting $\prec$ and $\models$ to $\mathcal{S}^*$, we obtain an extension.

We still have to make precise the concepts of "approaching" and "completion" by means of topology. Normally, differences between objects taken from a set M are measured by nonnegative real numbers, i.e., with a metric $d : M \times M \to \mathbb{R}^+$. For our purpose it seems to be more natural, to measure differences by sets of formulas, i.e., with a *pseudometric* $d : M \times M \to 2^{\mathcal{L}}$. That means, the target structure $(\mathbb{R}^+, \leq, +)$ in the original definition of a metric space has to be replaced by $(2^{\mathcal{L}}, \subseteq, \cup)$. We may read $d(m_1, m_2) \subset d(m_1, m_3)$ as "m_1 is closer by m_2 than by m_3".

Definition 4.2. *A structure* (M, L, d) *is said to be* a pseudometric space *iff the following three points are satisfied:*

1. *M, L are sets, L is at most countable.*
2. *$d : M \times M \to 2^L$ is a function, satisfying for every $m_1, m_2, m_3 \in M$ the conditions*

   reflexivity: $d(m, m) = \emptyset$

   symmetry: $d(m_1, m_2) = d(m_2, m_1)$

   triangle axiom: $d(m_1, m_2) \subseteq d(m_1, m_3) \cup d(m_3, m_2)$

d is called a pseudometric.

Definition 4.3. *Let (M, L, d) be a pseudometric space. A sequence $\sigma \in M^\omega$ is said to be a* Cauchy-sequence in M *iff* $\forall A \in L : \exists N \forall i, j \geq N : A \notin d(\sigma(i), \sigma(j))$. $M^* \subseteq M^\omega$ *denotes the set of* Cauchy-sequences in M. $m \in M$ *is called limit of a sequence* σ *iff* $\forall A \in L : \exists n \forall i \geq n : A \notin d(m, \sigma(i))$. (M, L, d) *is said to be complete iff every* Cauchy-sequence $\sigma \in M^*$ *has a limit in* M.

The set of Cauchy-sequences in M itself may be considered as a pseudometric space, if we define a function $d^* : M^* \times M^* \to 2^L$ as follows:

$$d^*(\rho, \tau) := \bigcup_{n \in \omega} \bigcap_{i \geq n} d(\rho(i), \tau(i))$$

Lemma 4.4. *Let* (M, L, d) *be a pseudometric space. Then* (M^*, L, d^*) *is a pseudometric space.*

Proof. We prove the triangle axiom of d^*, the proofs of reflexivity and symmetry are straightforward. Let $\sigma, \tau, \upsilon \in M^*$ be Cauchy-sequences in M and $A \in d^*(\sigma, \upsilon)$. Then a natural number n exists, such that for every $i, j \geq n$ the following two points are satisfied:

- $A \in d(\sigma(i), \upsilon(i))$.
- $A \notin d(\sigma(i), \sigma(j)) \cup d(\tau(i), \tau(j)) \cup d(\upsilon(i), \upsilon(j))$.

Suppose $A \notin d^*(\sigma, \tau) \cup d^*(\tau, \upsilon)$. Then we have natural numbers $l_1, l_2 \geq n$, such that $A \notin d(\sigma(l_1), \tau(l_1)) \cup d(\tau(l_2), \upsilon(l_2))$. The triangle axiom yields $A \notin d(\sigma(i), \sigma(l_1)) \cup d(\sigma(l_1), \tau(l_1)) \cup d(\tau(l_1), \tau(l_2)) \cup d(\tau(l_2), \upsilon(l_2)) \cup d(\upsilon(l_2), \upsilon(i)) \supseteq d(\sigma(i), \upsilon(i))$ for any $i \geq n$. This gives a contradiction. □

Proposition 4.5. *Let* (M, L, d) *be a pseudometric space. Then the pseudometric space* (M^*, L, d^*) *is complete.*

Proof. Let $\{A_1, A_2, ..\}$ be an enumeration of $\mathcal{L}$ and $(\sigma_k) \in (M^*)^*$ a Cauchy-sequence in (M^*, L, d^*). We construct a limit of (σ_k) as follows:

- $N_1 := \min \{n | \forall l, k \geq n : A_1 \notin d^*(\sigma_l(i), \sigma_k(j))\}$.
- $M_1 := \min \{n | \forall i, j \geq n : A_1 \notin d(\sigma_{N_1}(i), \sigma_{N_1}(j))\}$.
- $N_{s+1} := \min \{n | \forall l, k \geq n : \{A_1, .., A_{s+1}\} \cap d^*(\sigma_l(i), \sigma_k(j)) = \emptyset\}$.
- $M_{s+1} := M_s + \min\{n | \forall i, j \geq n : \{A_1, .., A_{s+1}\} \cap$
  $d(\sigma_{N_{s+1}}(i), \sigma_{N_{s+1}}(j)) = \emptyset\}$.

Since (σ_k) is a Cauchy-sequence in (M^*, L, d^*), the N_s exist; since every σ_k is a Cauchy-sequence in (M, L, d), the M_s exist. It follows further immediately from the definition, that the sequences (M_s) and (N_s) are monotonic. Consider the sequence $\sigma_D \in M^\omega$, defined by $\sigma_D(i) := \sigma_{N_i}(M_i)$.

First we show, that σ_D is a Cauchy-sequence in (M^*, L, d^*). Let $A_n \in L$ be an arbitrary element of L. Then $d(\sigma_D(i), \sigma_D(j)) = d\left(\sigma_{N_i}(M_i), \sigma_{N_j}(M_j)\right) \subseteq d(\sigma_{N_i}(M_i), \sigma_{N_i}(M_j)) \cup d\left(\sigma_{N_i}(M_j), \sigma_{N_j}(M_j)\right)$ implies $A_n \notin d\left(m_i^D, m_j^D\right)$ for every $i, j \geq n$.

Next we prove that σ_D is a limit of (σ_k). Let $A_n \in L$ be again an arbitrary element of L and $k \geq N_n$. Then we have for every $i \geq n + N_n$ and $j \geq M_n$ $d(\sigma_D(i), \sigma_k(i)) \subseteq d(\sigma_D(i), \sigma_{N_i}(i)) \cup d(\sigma_{N_i}(i), \sigma_k(i))$, and hence $A_n \notin d(\sigma_D(i), \sigma_k(i))$. This yields $A_n \notin d^*(\sigma_D, \sigma_k)$ and thereby σ_D is a limit of (σ_k). □

Definition 4.6. *Let* L, S *be sets and* $\models \subseteq S \times L$ *a binary relation. Then we call*

- $Th_{\models}(s) := \{A | s \models A\}$ *the* theory of $s \in S$ *and*
- $Th_s(\sigma) := \bigcup_{n \in \omega} \bigcap_{i \geq n} Th_{\models}(\sigma(i))$ *the* theory of a sequence $\sigma \in S^{\omega}$.

For a set L, let $\circ : 2^L \times 2^L \to 2^L$ denote the symmetric difference, i.e.,

$$L_1 \circ L_2 := (L_1 \cup L_2) \setminus (L_1 \cap L_2).$$

Example 4.1, continued Consider the metric space $(\mathcal{M}_0, \mathcal{L}_{CPL}, d)$ generated by $(\mathcal{L}_{CPL}, \mathcal{M}_0, \models)$ as defined in 4.1, with $d(m_1, m_2) := Th_{\models}(m_1) \circ Th_{\models}(m_2)$. Then $\sigma := (\mathcal{L}_{atom} \setminus \{p_{2i}\}_{i \leq n})_{n \in \omega} \in \mathcal{M}_0^{\omega}$ is a Cauchy-sequence. Let $A \in \mathcal{L}_{CPL}$ be an arbitrary wff and $p_{i_1}, .., p_{i_k}$ the finite set of variables occurring in A. Put $n \geq \max(\{i_1, .., i_k\})$, it follows $A \notin d(\sigma(j), \sigma(l))$ for every $j, l \geq n$ and hence σ is a Cauchy-sequence. But $(\mathcal{L}_{CPL}, \mathcal{M}_0, d)$ is not complete. Suppose, there is a limit $m_\infty \in \mathcal{M}_0$ of σ. Then follows from the definition, that $m_\infty \models p_{2i-1}$ and $m_\infty \not\models p_{2i}$, and thereby $p_{2i-1} \in m_\infty$ and $p_{2i} \notin m_\infty$. Hence m_∞ is neither finite nor infinite and it follows $m_\infty \notin \mathcal{M}_0$, in contradiction to the assumption.

Definition 4.7. *Let be L, S sets, with L countable, and $\models \subseteq S \times L$ a relation. Then (S, L, d), with $d : S \times S \to 2^L$ defined by $d(s_1, s_2) := Th_{\models}(s_1) \circ Th_{\models}(s_2)$, is a pseudometric space.*

Proof. Reflexivity and symmetry follow immediately from the definition, we show the satisfaction of the triangle axiom. Suppose $A \in Th_{\models}(s_1) \circ Th_{\models}(s_2)$. Then we can assume without loss of generality $A \in Th_{\models}(s_1)$ and $A \notin Th_{\models}(s_2)$. Let s_3 be an additional element of S. Then it holds either $A \in Th_{\models}(s_3)$ or $A \notin Th_{\models}(s_3)$. In the first case follows $A \in Th_{\models}(s_2) \circ Th_{\models}(s_3)$, in the second case $A \in Th_{\models}(s_1) \circ Th_{\models}(s_3)$. Thereby the triangle axiom is fulfilled. □

Lemma 4.8. *Let be L, S sets, with L countable, $\models \subseteq S \times L$ a binary relation, $d : S \times S \to 2^L$ as defined in 4.7 and $\sigma, \tau \in S^*$ Cauchy-sequences in (S, L, d). Then $d^*(\sigma, \tau) = \emptyset$ iff $Th_s(\sigma) = Th_s(\tau)$.*

Proof. ($\Rightarrow$) Suppose $d^*(\sigma, \tau) = \emptyset$, but there exists an A, such that $A \in Th_s(\sigma)$ and $A \notin Th_s(\tau)$. Then there is a number n, such that for every $i \geq n$ holds $\sigma(i) \models A$. Since $d^*(\sigma, \tau) = \emptyset$ there exists a number $k \geq n$, such that $A \notin d(\sigma(k), \tau(k))$. $A \notin Th_s(\tau)$ implies the existence of a $l \geq k$, such that $\tau(l) \not\models A$. Since τ is a Cauchy-sequence, we may further assume that $A \notin d(\tau(k), \tau(l))$. This yields $A \notin d(\sigma(k), \tau(l)) \subseteq d(\sigma(k), \tau(k)) \cup d(\tau(k), \tau(l))$ and finally $\tau(l) \not\models A$, in contradiction to the assumption.
($\Leftarrow$) We show the converse. Assume $Th_s(\sigma) = Th_s(\tau)$. Suppose, there exists an $A \in d^*(\sigma, \tau)$. From the definition of Cauchy-sequences follows the existence of a number n, such that $A \notin d(\sigma(k), \sigma(l)) \cup d(\tau(k), \tau(l))$ for every $k, l \geq n$. Since $A \in d^*(\sigma, \tau)$, n may be chosen large enough, such that $A \in d(\sigma(k), \tau(k))$ for every $k \geq n$. We have to distinguish between two cases:

Case 1: $\sigma(k) \models A$. Then we have $\sigma(l) \models A$ and $\tau(l) \not\models A$ for every $l \geq n$, and thereby $A \in Th_s(\sigma)$ and $A \notin Th_s(\tau)$, in contradiction to the assumption.
Case 2: $\sigma(k) \not\models A$. Then holds $\sigma(l) \not\models A$ and thereby $\tau(l) \models A$ for every $l \geq n$, hence $A \notin Th_s(\sigma)$ and $A \in Th(\tau)$, in contradiction to $Th_s(\sigma) = Th_s(\tau)$.
Putting the two cases together, $d^*(\sigma, \tau) = \emptyset$ is proved. □

It follows from the last lemma, that the theory of a class of Cauchy-sequences modulo the relation $d^*(\sigma, \tau) = \emptyset$ can be defined independently from the choice of an representative by $Th_s([\sigma]) := Th_s(\sigma)$.

5. Topological compactness

The representation theorems for (finitary) strongly cumulative inference operations due to MAKINSON [Mak89], KRAUS/LEHMANN/MAGIDOR [KLM90] and FREUND/LEHMANN [FL94] show that $C(X)$ ($F(X)$, respectively) can be retrieved from $\{C(Y) | Y \subseteq \mathcal{L}\}$ ($\{F(Y) | Y \subseteq_f \mathcal{L}\}$). Put

- $\mathcal{S} := \{F(X) | X \subseteq_f \mathcal{L}\}$
- $F(X) \models A$ iff $A \in F(X)$
- $F(X) \prec_0 F(Y)$ iff $\exists X' \subseteq_f \mathcal{L} : F(X) = F(X') \& X' \subseteq F(Y) \& F(X) \neq F(Y)$
- $\prec$ is the transitive closure of $\prec_0$

Then $F(X)$ is precisely the $\prec$ −minimal element of $\{s \in \mathcal{S} | \forall A \in X : s \models A\}$. The structure $\mathbf{S}(F) := (\mathcal{L}, \mathcal{S}, \prec, \models)$ is called the *canonical model* of the finitary cumulative inference operation F. In particular, strong cumulativity yields, that $\prec$ is irreflexive and antisymmetric. It follows from the last section, that $\mathbf{S}(F)$ can be considered as a pseudometric space. The idea of the topological extension is as follows: first to complete this space to $\mathcal{S}^*$, next to adapt the relations $\prec$ and $\models$ to $\mathcal{S}^*$, and finally to define $C(X)$ for arbitrary sets X from the completed model. More precisely, $\prec$ and $\models$ are adapted as follows:

- $d(F(X), F(Y)) := F(X) \circ F(Y)$
- $\mathcal{S}^*$ is the set of Cauchy sequences of the pseudometric space $(\mathcal{L}, \mathcal{S}, d)$
- $\sigma \models^* A$ iff $A \in Th_s(\sigma)$
- $\sigma \preceq^* \tau$ iff $\exists n \forall i \geq n : \sigma(i) \preceq \tau(i)$.

$\preceq^*$ is transitive, but not irreflexive and antisymmetric yet. In order to ensure this property, we have to factorize $(\mathcal{L}, \mathcal{S}^*, \prec^*, \models^*)$ modulo the relation $\sim \subseteq \mathcal{S}^* \times \mathcal{S}^*$, defined by $\sigma \sim \tau$ iff $\exists n \forall i \geq n : \sigma(i) = \tau(i)$. By $\sim$, sequences with common end sequences are getting identified. It follows immediately that $\sim$ is an equivalence relation and that the following definitions of $\models^+$ and $\prec^+$, respectively, are independent of the choice of a representative.

- $\mathcal{S}^+ := \mathcal{S}^* / \sim$
- $[\sigma]_\sim \models^+ A$ iff $A \in Th_s(\sigma)$
- $[\sigma]_\sim \prec^+ [\tau]_\sim$ iff $[\sigma]_\sim \neq [\tau]_\sim$ and $\sigma \preceq^* \tau$

Then $\prec^+$ is transitive, irreflexive and antisymmetric. $\mathbf{S}^+(F) := (\mathcal{L}, \mathcal{S}^+, \prec^+, \models^+)$ is said to be the *completion* of $\mathbf{S}(F)$.

$\hat{X} := \left\{ [\tau]_\sim | \forall B \in X : [\tau]_\sim \models^+ B \right\}$ denotes the class of elements from $\mathcal{S}^+$ which verify X.

Definition 5.1. *Let $F : 2^{\mathcal{L}}_f \to 2^{\mathcal{L}}$ be a strongly cumulative finitary inference operation, $\mathbf{S}(F) := (\mathcal{L}, \mathcal{S}, \prec, \models)$ its canonical model and $\mathbf{S}^+(F) := (\mathcal{L}, \mathcal{S}^+, \prec^+, \models^+)$ as defined above. The extension $\Delta_{top} F$ is defined as follows: $A \in \Delta_{top} F(X)$ iff $[\sigma]_\sim \models^+ A$ for every $[\sigma]_\sim$ minimal with respect to $\prec^+$ in $\hat{X}$.*

Proposition 5.2. *Let $F : 2^{\mathcal{L}}_f \to 2^{\mathcal{L}}$ be a strongly cumulative finitary inference operation. Then Δ_{top} is indeed an extension, i.e., $F = (\Delta_{top} F)_f$.*

Proof. Let $\mathbf{S}^+(F) := (\mathcal{L}, \mathcal{S}^+, \prec^+, \models^+)$ be the completion of the canonical model of $\mathbf{S}^+(F)$ and $X_0 \subseteq_f \mathcal{L}$ be a finite set. A state $[\sigma]_\sim$ is said to be *quasi stationary* iff there exists a representative $\sigma \in [\sigma]_\sim$, satisfying the condition $\exists n \forall i, j \geq n : \sigma(i) = \sigma(j)$. We show that the quasi stationary elements of the type $\sigma_0 := (F(X_1), .., F(X_n), F(X_0), F(X_0), ..)$ are precisely the $\prec^+$-minimal elements of $\hat{X}_0$. Suppose $[F(Y_i)] \in \hat{X}_0$. By finiteness of X_0, there exists a number m, such that for every $i \geq m$ holds $X_0 \subseteq F(Y_i)$. It follows $\forall i > m + n$: $F(X_0) \preceq F(Y_i)$ for every $i > m + n$ and thereby $\sigma_0 \preceq^* F(Y_i)$. Since the theory of σ_0 is $F(X_0)$, it follows $\Delta_{top} F(X_0) = F(X_0)$. □

Hence the operator Δ_{top} defined by

$$\Delta_{top}(\mathcal{L}, Cn, F) := (\mathcal{L}, Cn, \Delta_{top} F)$$

is an extension on the class of cumulative finitary inference frames. Note that, in contrast to $\Delta_1 - \Delta_4$, $\Delta_{top} F$ does not depend on the base Cn of the frame. A completely Δ_{top}-compact inference frame is also said to be *topologically compact.*

Theorem 5.3. *Δ_{top} is conservative.*

Proof. Let Cn be a deductive operation, $A_1, .., A_n, ..$ an enumeration of X and $\sigma := (Cn(X_i))$ a Cauchy-sequence, such that $[\sigma]_\sim \models A$ for every $A \in X$. Then there are natural numbers $N(1) < N(2) < ..$, such that $\{A_1, .., A_k\} \subseteq Cn(X_i)$ for any $i \geq N(k)$. Consider the sequence τ:

$$\tau = \left(\underbrace{\emptyset, .., \emptyset}_{(N(1)-1)-times}, .., \underbrace{Cn(Y_1), .., Cn(Y_1)}_{(N(2)-N(1))-times}, .., \underbrace{Cn(Y_k), .., Cn(Y_k)}_{(N(k+1)-N(k))-times}, .. \right),$$

with $Y_k := \{A_1, .., A_k\}$. It follows by monotony and compactness of Cn, that τ is a Cauchy-sequence, $Th_s(\tau) = Cn(X)$ and $\tau \preceq^* \sigma$. Note that τ depends on both the enumeration and the choice of the numbers $N(k)$. Hence not only τ, but the set of sequences differing from τ only with respect to the choice of the enumeration and the choice of the sequence $(N(k))$ as well are precisely the minimal elements of $\hat{X}$. It follows $\Delta_{top}Cn_f = Cn$ and topological compactness is proved.

We show the converse, suppose $\Delta_{top}Cn_f := Cn$, and let X be a set. Then it holds $Th_s(\tau) \supseteq Cn(X)$ for every minimal element $[\sigma]_\sim$ in $\hat{X}$. As in the first part, the minimal elements are precisely of the type τ. Since we do not have compactness of Cn yet, we have only $Th_s(\tau) \subseteq Cn(X)$. Putting these together, it follows $Th_s(\tau) = Cn(X)$. Consider a wff $A \in Cn(X)$. Then there exist a number n, such that $A \in \tau(i)$ for any $i \geq n$ and thereby $A \in Cn(Y)$ for a finite subset Y of X. □

Lemma 5.4. *Let $\mathcal{IF} := (\mathcal{L}, Cn, F)$ be a strongly cumulative finitary inference frame and σ a Cauchy-sequence in $\mathbf{S}(F)$. Then $Th_s(\sigma)$ is a Cn-theory, i.e. $Th_s(\sigma) = Cn\,(Th_s(\sigma))$.*

Proof. Suppose $A \in Cn\,(Th_s(\sigma))$. By compactness of Cn, there exists a finite set $Y \subseteq_f Th_s(\sigma)$, such that $A \in Cn(Y)$. By definition of Th_s, there is a number n, such that $Y \subseteq \sigma(i) := F(X_i)$ for every $i \geq n$. Left absorbing yields $A \in CnF(X_i) = F(X_i)$ for every $i \geq n$ and thereby $A \in Th_s(\sigma)$. □

Theorem 5.5. *MRPL is topologically compact.*

Proof. Let $p_1, p_2, ..$ be an enumeration of $\mathcal{L}_{atom}$ and $H^X_{V_n}$ as defined in the proof of proposition 3. First we show, that

$$\sigma := \left(C_{\min}\left(\left\{H^X_{V_n}\right\}\right)\right)$$

is a Cauchy-sequence. There are two cases.

Case 1: $A \in C_{\min}(X)$. We can choose n large enough, such that each variable of A is in V_n. As shown in the proof of proposition 3, we have $A \in C_{\min}\left(\left\{H^X_{V_i}\right\}\right)$ for every $i \geq n$, and thereby $A \notin d\left(C_{\min}\left(\left\{H^X_{V_n}\right\}\right), C_{\min}\left(\left\{H^X_{V_n}\right\}\right)\right)$ for every $i, j \geq n$.

Case 2: $A \notin C_{\min}(X)$.

Let $X \subseteq \mathcal{L}$ a set of wffs and $A_1, A_2, ..$ an enumeration of $C(X)$. We construct a sequence

A Δ_4-sequence $\sigma \in \{C(Y) | Y \subseteq_{finite} \mathcal{L}\}^\omega$ is defined as follows:

- $(A_k) \in (C(X))^\omega$ an enumeration of $C(X)$.
- $(B_k) \in (C(X))^\omega$ is defined by $B_k := A_1 \wedge .. \wedge A_k$
- $(X_k) \in \left(2_f^{Cn(X)}\right)^\omega$ is an increasing sequence of finite subsets of $Cn(X)$ satisfying the conditions $Cn(X) := \bigcup X_i$ and $B_k \in C\left(\{X_k\}\right)$ for every k.

The existence of the sequences (X_k) is ensured by complete Δ_4-compactness: for $k := 1$ there exists a finite set $X_1 \subseteq Cn(X)$ such that $A_1 = B_1 \in C(X_1)$. For a given X_k a successor X_{k+1} can be constructed as follows: by assumption, there is a finite set $X'_{k+1} \subseteq Cn(X)$, such that $B_{k+1} \in C(X_k \cup X'_{k+1})$. Put $X_{k+1} := X_k \cup X'_{k+1}$. Note that X'_{k+1} can be chosen as large as necessary in order to ensure $Cn(X) := \bigcup X_i$. The conjunction property implies $\{A_1, .., A_k\} \subseteq C(X_k)$. Finally, σ is defined by

$$\sigma. = \left(\underbrace{C(X_1), .., C(X_1)}_{N_1 - times}, .., \underbrace{C(X_k), .., C(X_k)}_{N_k - times}, .. \right),$$

i.e., each set $C(X_k)$ may be repeated finitely many times. Then complete Δ_4-compactness implies $Th_s(\sigma) = C(X)$ for every Δ_4-sequence. Since the construction of σ depends on the choice of an enumeration, there may exist an infinite number of Δ_4-sequences for a given set X.

We show that the classes modulo $\sim$ of Δ_4-sequences $[\sigma]_\sim$ are precisely the minimal elements in $\hat{X}$ in $\mathbf{S}^+(C_f)$. $[\sigma]_\sim \in \hat{X}$ follows from $Th_s(\sigma) = C(X)$. Let $[\tau]_\sim \in \hat{X}$ be an arbitrary sequence in $\hat{X}$, $\tau := (C(Y_i))$ and (X_k) a sequence of finite subsets of $Cn(X)$ as defined above. From lemma 5.4 follows, that $Cn(X) \subseteq Th_s(\tau)$ and thereby $\sigma(i) \subseteq Th_s(\tau)$ for every member $\sigma(i)$ of a Δ_4-sequence σ. Hence there exists a strictly increasing sequence of natural numbers $N(1) < N(2) < ..$ satisfying the condition $\forall i \geq N(k) : X_k \subseteq Th_s(\tau)$. Consider the sequence σ'_τ defined by

$$\sigma'_\tau = \left(\underbrace{\emptyset, .., \emptyset,}_{(N(1)-1)-times} \; .., \underbrace{C(X_1), .., C(X_1)}_{(N(2)-N(1))-times}, .., \; \underbrace{C(X_k), .., C(X_k)}_{(N(k+1)-N(k))-times} \; , .. \right).$$

By definition of $\preceq^*$ follows $\sigma'_\tau \preceq^* \tau$ and therefore either $[\sigma'_\tau]_\sim \prec^+ [\tau]_\sim$ or $[\sigma'_\tau]_\sim = [\tau]_\sim$. If we replace the first $N(1) - 1$ elements of σ'_τ by $C(X_1)$, we get a Δ_4-sequence σ_τ. I follows straightforward, that $[\sigma'_\tau]_\sim = [\sigma_\tau]_\sim$.

Putting these together, it is proved that $\mathbf{S}^+(C_f)$ is stoppered, and that the minimal elements in $\hat{X}$ are precisely the classes modulo $\sim$ of Δ_4-sequences, and hence $\mathcal{IF}$ is topologically compact. □

Corollary 5.6. *Minimal reasoning in classical propositional logic is topologically compact.*

Corollary 5.7. *Topological compactness satisfies the strong principle of weakness.*

Proof. The proof can be derived straightforward from the weak compactness of MRPL. If we replace the basis of this frame by the identity $id : 2^{\mathcal{L}_{CPL}} \to 2^{\mathcal{L}_{CPL}}$, $id(X) := X$, then we get the frame $\mathcal{IF}' := (\mathcal{L}_{CPL}, id, C_{\min})$. Then $\mathcal{IF}'$ is not weakly compact, since weak compactness of $\mathcal{IF}'$ is equivalent to compactness of C. Further it follows from the topological compactness of $\mathcal{IF}$, that $\mathcal{IF}$ is topologically compact too: the definition of Δ_{top} does not depend on the choice of a base, in contrast to the definitions of $\Delta_1, .., \Delta_4$. □

From the last results follows that topological compactness does not share the drawbacks of the canonical extension: not weakly compact inference frames may be handled. It remains an open problem, whether some important frames like minimal reasoning in first order logic and general POOLE-systems are topologically compact. But there seems to be a price to pay: it is not clear, whether Δ_{top} preserves properties like strong cumulativity and distributivity, i.e., whether the satisfaction of the finitary version of a condition by F implies the satisfaction of the condition by $\Delta_{top}F$.

Proposition 5.8. *Let $F : 2^{\mathcal{L}}_f \to 2^{\mathcal{L}}$ be a strongly cumulative finitary inference operation and $\mathbf{S}(F)$ its canonical model. Then $\mathbf{S}^+(F)$ is stoppered, i.e., for any set $X \subseteq \mathcal{L}$ and any $[\tau]_\sim \in \hat{X}$ the following holds: $[\tau]_\sim$ is either minimal with respect to $\prec^+$ in $\hat{X}$ or there exists a $[\sigma]_\sim$ minimal in $\hat{X}$ such that $[\sigma]_\sim \prec^+ [\tau]_\sim$.*

Proof. Suppose there exists a $[\tau_0]_\sim \in \hat{X}$ which is neither minimal in $\hat{X}$ nor an extension with respect to $\prec^+$of a minimal element in $\hat{X}$. This yields the existence of an infinite chain $\Gamma_0^+ := [\tau_0]_\sim \succ^+ [\tau_1]_\sim \succ^+ [\tau_2]_\sim ..$ in $\hat{X}$. Put $\tau_k := \left(F(X_i^k)\right)$, and let $A_1, A_2, ..$ be an enumeration of $\mathcal{L}$. Further, let $\tau_0, \tau_1, \tau_2, ..$ a collection of representatives of Γ_0^+. We define a set of representatives $\tau'_0, \tau'_1, \tau'_2, ..$ recursively as follows:

- $\tau'_0 := \tau_0$
- by definition of $\succ^+$, there exists an index $N(1)$, such that $\tau_0(i) \preceq \tau_1(i)$ for any $i \geq N(1)$. Put

  $$\tau'_1(i) := \begin{cases} F(\emptyset) & \text{if} \quad i < N(1) \\ \tau_1(i) & \text{if} \quad i \geq N(1) \end{cases}$$

- let $N(k+1)$ a number, such that $\tau'_j(i) \preceq \tau_{k+1}(i)$ for any $j \leq k$ and any $i \geq N(k+1)$. Put

  $$\tau'_{k+1}(i) := \begin{cases} F(\emptyset) & \text{if} \quad i < N(k+1) \\ \tau_{k+1}(i) & \text{if} \quad i \geq N(k+1) \end{cases}$$

It follows immediately from the definition, that the τ'_k 's are indeed representatives of the classes $[\tau_2]_\sim$. Since $F(\emptyset) \preceq F(Y)$ for any Y, we have $\tau'_k \preceq^* \tau'_l$ iff $\tau'_k(i) \preceq \tau'_l(i)$ for *any* i.

The sequence $\sigma \in \mathcal{S}^\omega$ is defined as follows:

step 1 *case 1:* $[\tau'_i]_\sim \models^+ A_1$ for an infinite number of indices i. Then Γ_1 is obtained from Γ_0 by removing every τ'_j, such that $\left[\tau'_j\right]_\sim \not\models^+ A_1$. Further put $Y_1^+ := \{A_1\}$ and $Y_1^- := \emptyset$.

case 2: Otherwise Γ_1 is obtained from Γ_0 by removing every τ'_j, such that $\left[\tau'_j\right]_\sim \models^+ A_1$. In this case put $Y_1^+ := \emptyset$ and $Y_1^- := \{A_1\}$.

Let $\tau'_{n(1)}$ be the first element of Γ_1 and let $m(1)$ by a number, such that $\tau'_{n(1)}(i) \models A_1$ for any $i \geq m(1)$ in the first and $\tau'_{n(1)}(i) \not\models A_1$ for any $i \geq m(1)$ in the second case.

step k+1 *case 1:* $[\tau'_i]_\sim \models^+ A_{k+1}$ for an infinite number of members of Γ_k. In this case we obtain Γ_{k+1} by removing every τ'_j, such that $\left[\tau'_j\right]_\sim \not\models^+ A_{k+1}$ from Γ_k. Put $Y_{k+1}^+ := Y_k^+ \cup \{A_{k+1}\}$ and $Y_{k+1}^- := Y_k^-$.

case 2: Otherwise Γ_{k+1} is obtained from Γ_k by removing every τ'_j, such that $\left[\tau'_j\right]_\sim \models^+ A_{k+1}$. Put $Y_{k+1}^+ := Y_k^+$ and $Y_{k+1}^- := Y_k^- \cup \{A_{k+1}\}$.

Let $n(k+1)$ be the least number, such that $n(k) < n(k+1)$ and $\tau'_{n(k+1)}$ is in Γ_{k+1}. $m(k+1)$ is defined to be the least number, such that for any $i \geq m(k+1)$ the following conditions are satisfied:

- $\tau'_{n(k+1)}(i) \models A_j$ for any $A_j \in Y_{k+1}^+$
- $\tau'_{n(k+1)}(i) \not\models A_j$ for any $A_j \in Y_{k+1}^-$

By definition of $\mathbf{S}^+(F)$ and by the Cauchy-property of the τ_j's, such an index $m(k+1)$ exists. Finally we define

$$\sigma(i) := \begin{cases} F(\emptyset) & \text{if} \quad i < m(1) \\ F\left(X_{m(k)}^{n(k)}\right) & \text{if} \quad m(k) \leq i < m(k+1) \end{cases} .$$

It follows immediately from the definition, that σ is a Cauchy-sequence: for any wff $A_k \in \mathcal{L}$ we have either $A_k \in F\left(X_{m(i)}^{n(i)}\right)$ for any $i > k$ or $A_k \notin F\left(X_{m(i)}^{n(i)}\right)$ for any $i > k$. In particular, the elements of X are of the first type and it follows $\sigma \in \hat{X}$. Consider a fixed τ'_i. Since $n(1) < n(2) < ..$ is strictly increasing, there is a number k, such that $i < n(k)$. By the way $m(k)$ was defined, we have $\tau'_i(j) \succeq \tau'_{n(k)}(j)$ for any $j \geq m(k)$. Further we have $\tau'_{n(k)} \succeq^* \sigma$ and, by transitivity[5] of $\succeq$ and $\succeq^*$, repectively; it follows that $\tau'_i \succeq^* \sigma$.

Putting these together, σ is a lower bound of $\{\tau'_0, \tau'_1, \tau'_2, ..\}$ (with respect to $\preceq^*$). Since the definition of $\prec^*$does not on the representatives used, $[\sigma]_\sim$ is a lower bound of Γ_0^+ with respect to $\prec^+$. By ZORNs lemma we conclude that any element in $\hat{X}$ is either minimal or an extension (with respect to $\prec^+$) of a minimal element. Thereby $\mathbf{S}^+(F)$ is stoppered.

Figure 5.1 shows the construction of σ, the rows represent the sequences τ'_i, the entries in the last column show the "stabilized" formulas, i.e., formulas A, such that either $\tau \models^* A$ for *every* following row with an index of the type $N(i)$ or $\tau \not\models^* A$ for *every* following row with an index of this type. By the choice of representatives and by the way σ was defined, any sequence $F\left(X_i^1\right)$, (FX_i^2),..,$\sigma(i)$ of the type $\omega + 1$ is monotonic with respect to $\succeq$.

[5]At this point, transitivity of $\preceq$ and hence *strong* cumulativity of F is needed.

τ'_1	$F(X_1^1)$	..	$F\left(X^1_{m(1)}\right)$	..		..	
τ'_2	$F(\emptyset)$	..	$F\left(X^2_{m(1)}\right)$	..		..	
$\vdots$				..		..	
$\tau'_{n(1)}$	$F(\emptyset)$	..	$F\left(X^{n(1)}_{m(1)}\right)$	..		..	A_1
$\vdots$				..		..	
$\tau'_{n(2)}$	$F(\emptyset)$	..	$F\left(X^{n(2)}_{m(1)}\right)$	..	$F\left(X^{n(2)}_{m(2)}\right)$	..	A_1, A_2
$\vdots$							
σ	$F(\emptyset)$	..	$F\left(X^{n(1)}_{m(1)}\right)$	..	$F\left(X^{n(2)}_{m(2)}\right)$	..	$A_1, A_2, A_3, ..$

Table 5.1: On the construction of σ

Corollary 5.9. *If $F : 2^{\mathcal{L}}_f \to 2^{\mathcal{L}}$ is a strongly cumulative finitary inference operation, then $\Delta_{top}F : 2^{\mathcal{L}} \to 2^{\mathcal{L}}$ is strongly cumulative.*

Proof. Straight forward from the last proposition and a result from MAKINSON [Mak89].□

6. Conclusion and open problems

Table 6.1 evaluates the extensions on the base of the claims of conservativeness, weakness, ultra weakness and preserving strong cumulativity. Conservativeness of $\Delta_2 - \Delta_4$ follows immediately from the definitions. Whether $\Delta_2 - \Delta_4$ preserve strong cumulativity is not known to us.

condition	Δ_1	Δ_2	Δ_3	Δ_4	Δ_{top}
conservative	no	yes	yes	yes	yes
weak	no	no	no	yes	yes
ultra-weak	no	no	no	no	yes
preserves loop	yes	?	?	?	yes

Table 6.1: Extensions for strongly cumulative inferential frames

Although we have provided an example which shows that topological compactness is ultra weak, we do not know yet, whether the important inferential frames (general POOLE-systems, minimal reasoning in first order logic) which are known to be not weakly compact, are indeed topologically compact.

An advantage of Δ_{top} is its flexibility: its definition depends on the choice of a metric d. But the framework set up above allows to replace the symmetric difference by any other metric. For instance, given a metric space (M, L, d) and a subset $L' \subseteq L$, the function $d_{L'} : M \times M \to 2^L$, defined by

$$d_{L'}(m_1, m_2) := L' \cap d(m_1, m_2),$$

satisfies reflexivity, symmetry and the triangle axiom, i.e., $(M, L, d_{L'})$ is a metric space too. The investigation of the extensions resulting from alternative metrics is another task for further investigations.

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