# From Partial to Possibilistic Logic

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**Abstract.** Based on our earlier work on partial logics and extended logic programs [Wag91, Wag94, HJW96], and on the possibilistic logic of [DLP94], we define a compositional possibilistic first-order logic with two kinds of negation.

## 1 Introduction

In this paper, we first define *fuzzy* and *possibilistic* Herbrand interpretations as conservative extensions of classical and partial Herbrand interpretations, and then define corresponding satisfaction relations for uncertainty-qualified sentences as the semantical basis of our (semi-)possibilistic logic. Unlike the *possibility theory* of Zadeh, Dubois and Prade,[1] our possibilistic logic supports, like partial logic, two kinds of negation.

The following examples of (heuristic) deduction rules may motivate the need for two kinds of negation in uncertain reasoning. First, consider the MYCIN-like rule

classification( Organism, bacteroides)
$\overset{0.5}{\longleftarrow}$ gram_stain( Organism, gram_negative),
morphology( Organism, rod),
$\sim$ aerobic( Organism).

Here, the negative condition $\sim$*aerobic*(*Organism*) requires definite negative evidence, i.e. evidence that the *Organism* is anaerobic, while in the following rule about the diagnosis of hepatitis only the absence of positive evidence is required by the negated conditions (similar to negation-as-failure):

diagnosis( Patient, hepatitis_X)
$\longleftarrow$ diagnostic_finding( Patient, cirrhosis),
$\neg$ diagnosis( Patient, hepatitis_A),
$\neg$ diagnosis( Patient, hepatitis_B),
...
$\neg$ diagnosis( Patient, hepatitis_G).

[1] See [Zad79, DP80, DLP94]. In contrast to this classical work, where the fundamental notion is the non-logical concept of a *possibility distribution*, we reconstruct fuzzy and possibilistic logic on the basis of the two logically fundamental notions of an interpretation and a satisfaction relation.

This rule expresses the heuristic, that if there is evidence for the diagnostic finding *cirrhosis* (say, its certainty is 0.8), and there is no evidence for a diagnosis of either of hepatitis A to G, then the diagnosis is hepatitis X (with certainty 0.8).

In real world knowledge bases like, for instance, relational or deductive databases, it is essential to be able to infer negative information by means of *minimal* (or *stable*) entailment, i.e. drawing inferences on the basis of minimal (or stable) models. These notions require an information ordering between models. The same situation arises in the framework of uncertain reasoning with fuzzy and possibilistic databases.

## 2 Preliminaries

A *signature* $\sigma = \langle Rel, Const, Fun \rangle$ consists of a set of relation symbols, a set of constant symbols, and a set of function symbols. $U_\sigma$ denotes the set of all ground terms of $\sigma$. For a term tuple $t_1, \ldots, t_n$ we also write $\mathbf{t}$ when its length is of no relevance. The logical functors are $\neg, \sim, \wedge, \vee$; where $\neg$ and $\sim$ are called *weak* and *strong* negation. $L(\sigma)$ is the smallest set containing the atomic formulas of $\sigma$, and being closed with respect to the following conditions: if $F, G \in L(\sigma)$, then $\{\neg F, \sim F, F \wedge G, F \vee G\} \subseteq L(\sigma)$. For sublanguages of $L(\sigma)$ formed by means of a subset $\mathcal{F}$ of the logical functors, we write $L(\sigma; \mathcal{F})$. With respect to a signature $\sigma$ we define the following sublanguages: $\mathrm{At}(\sigma) = L(\sigma; \emptyset)$, the set of all atomic formulas (also called *atoms*); $\mathrm{Lit}(\sigma) = L(\sigma; \sim)$, the set of all *literals*; $\mathrm{XLit}(\sigma) = \mathrm{Lit}(\sigma) \cup \{\neg l : l \in \mathrm{Lit}(\sigma)\}$, the set of all *extended literals*, and finally $L(\sigma, +) = L(\sigma; \neg, \wedge, \vee)$, the set of formulas without strong negation. We introduce the following conventions. When $L \subseteq L(\sigma)$ is some sublanguage, $L^0$ denotes the corresponding set of sentences (closed formulas). If the signature $\sigma$ does not matter, we omit it and write, e.g., $L$ instead of $L(\sigma)$.

If $Y$ is a preorder, then $\mathrm{Min}(Y)$ denotes the set of all minimal elements of $Y$, i.e. $\mathrm{Min}(Y) = \{X \in Y \mid \neg \exists X' \in Y : X' < X\}$.

**Definition 1 (Certainty Scale)** *A certainty scale* $\langle \boldsymbol{C}, 0, 1 \rangle$ *is a linearly ordered set* $\boldsymbol{C}$ *with least and greatest elements* 0 *and* 1.[2]

Examples of certainty scales are the rational unit interval, or any discrete ordering of linguistic uncertainty values such as $\langle 0, ll, ql, vl, 1 \rangle$, where $ll$ stands for *little likely*, $ql$ for *quite likely*, and $vl$ for *very likely*. In the sequel, we assume that there is a fixed certainty scale $\boldsymbol{C}$ for which we simply write $[0, 1]$. If we want to exclude the value for complete uncertainty, we write $(0, 1] = \{v \in \boldsymbol{C} \mid v > 0\}$.

[2] From a purely formal point of view, one could as well choose the more general structure of a (distributive) lattice of certainty values instead of a linear order. But from a practical point of view, I don't see any need for such a generalization.

## 3 The Logical Semantics Problem of Reasoning with Uncertainty

In [Elk93], Elkan argues that the semantic principles, on which fuzzy logic is based, are inconsistent in the sense that if they are taken seriously, the respective formal system collapses to 2-valued logic, i.e. does not allow for gradual uncertainty. These principles, according to Elkan, consist of the three evaluation functions min, max and probabilistic complement for conjunction, disjunction and negation, and the postulate that logically equivalent sentences are equally evaluated. Formally,

1. $v(F \wedge G) = \min(v(F), v(G))$
2. $v(F \vee G) = \max(v(F), v(G))$
3. $v(\neg F) = 1 - v(F)$
4. $v(F) = v(G)$ iff $F \equiv G$.

Elkan shows that if $\equiv$ is taken to mean logical equivalence in classical logic, then for any two sentences $F$ and $G$, either $v(F) = v(G)$, or $v(F) = 1 - v(G)$.

In their answer to Elkan, Dubois and Prade [DP94] point out that this result was already established in [Wes87] and [DP88]. They argue, however, that fuzzy logic is not based on classical equivalence (requiring $[0,1]$ to be a Boolean algebra), but rather defines its own notion of logical equivalence (based on the DeMorgan algebra $[0,1]$). Therefore, principle (4) cannot be a postulate refering to classical logic, but only a definition of equivalence in fuzzy logic.

Clearly, fuzzy logic violates both the *law of the excluded middle (LEM)*,

$$\text{(LEM)} \quad v(F \vee \neg F) = 1,$$

and the *law of the excluded contradiction (LEC)*,

$$\text{(LEC)} \quad v(F \wedge \neg F) = 0.$$

Especially the violation of LEC casts doubt on the logical justification of fuzzy logic.

### 3.1 A Natural Solution

Elkan has anticipated the answer of Dubois and Prade to some degree, since he discusses possible alternatives to classical equivalence for principle (4). He admits the possibility that LEM is not valid in uncertain reasoning, and the resulting logic, therefore, is subclassical. He seems, however, not prepared to accept the invalidity of its dual, viz. LEC. He says that

> One could hope that fuzzy logic is therefore a formal system whose tautologies are a subset of the classical tautologies, and a superset of the intuitionistic tautologies. [...] It is an open question how to choose a notion of logical equivalence that simultaneously (i) remains philosophically justifiable, (ii) allows useful inferences in practice, and (iii) removes the opportunity to prove results similar to Theorem 1 [the above trivialization result].

Systems between intuitionistic and classical logic are called *intermediate*.[3] In intermediate logics, LEM does not hold, but LEC does. Since both LEM and LEC are the basic principles of classical logic, this means that intermediate (i.e. *supraintuitionistic*) logics are closer to classical logic, and philosophically better justified, than the weak logics arising from DeMorgan algebras based on principles (1)-(3).

On the one hand side, Elkan is right in his quest for an intermediate logic as the logical basis of reasoning with uncertainty. On the other hand side, he seems to overlook the fact that principle (3), i.e. the fuzzy logic evaluation of negation, is not compatible with intuitionistc logic (it is not a Heyting complement). In fact, it is this principle, i.e. Zadeh's definition of negation, which is the culprit in the violation of LEC. A solution to the logical semantics problem following the above suggestions of Elkan requires therefore to redefine the evaluation of negation in the following way:

$$(3') \quad v(\neg F) = *v(F),$$

where the complement operation $*$ is defined as

$$*x = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ otherwise} \end{cases}$$

This leads to the (implication-free) Heyting algebra $\langle [0,1], *, \min, \max \rangle$, and the resulting intermediate logic is the system generated by all linear Heyting algebras which is also obtained by adding the axiom $(F \to G) \vee (G \to F)$ to the set of intuitionistic axioms. This logic is 'pretty close' to classical logic,[4] and it allows to evaluate sentences in arbitrary linear orderings with least and greatest elements 0 and 1.

Below, we will define this logic under the name *semi-possibilistic logic* since we consider it as a fragment of our compositional possibilistic logic (with two kinds of negation) in the same sense as classical logic can be considered as a fragment of partial logic with two kinds of negation (see [HJW96]).

### 3.2 The Non-Compositional Possibility-Theoretic Approach

Dubois and Prade [1994] argue that in order to retain classical tautologies, i.e. the classical logic reading of principle (4), one has to sacrifice compositionality and give up some of the sentence evaluation functions (1)-(3). In fact, in their non-compositional possiblistic logic [DLP94], neither (2) nor (3) is valid. While (2) is replaced by the inequality

$$(2') \quad v(F \vee G) \geq \max(v(F), v(G))$$

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[3] There are exactly five proper intermediate logics which are well-behaved, i.e. which have the fundamental *interpolation* property, see [Mak77].

[4] In fact, a weak version of LEM holds: $v(\neg F \vee \neg\neg F) = 1$.

allowing for $v(F \vee \neg F) = 1$, yet $\max(v(F), v(\neg F)) < 1$, negation is evaluated by means of a dual measure $\tilde{v}$ according to

$$(3'') \quad v(\neg F) = 1 - \tilde{v}(F)$$

such that (1),(2′), and (3″) are compatible with the classical logic reading of postulate (4).

While (3″) is a well-justifiable choice of evaluating negation, it seems questionable to give up compositionality. In logic, unlike in probability theory, compositionality is one of the most basic principles (not only since Frege). It is much more fundamental than the principle of bivalence constituting classical logic. From a logical point of view, therefore, it does not seem to be justified to give up the more fundamental principle of compositionality in favor of the less fundamental principle of bivalence (i.e. classical tautologies). The non-compositional possibilistic logic of [DLP94], although it includes (as opposed to fuzzy logic) a well-justified treatment of negation, cannot be considered a good solution to the logical semantics problem of reasoning with uncertainty-qualified sentences.[5]

In our compositional possibilistic logic, we combine the dualistic treatment of negation of [DLP94] with the supraintuitionistic negation proposed above, resulting in a system with two kinds of negation expressing two different notions of falsity, like partial logic with two kinds of negation.

## 4 Semi-Possibilistic Logic

It is important to note that in (semi-)possibilistic logic, unlike in probability theory, the intuitive meaning of 0 is not *false*, or impossible, but rather *completely uncertain*, or absolutely *no information*.

A *fuzzy Herbrand interpretation* of the language $L(\sigma)$ is based on fuzzy relations over the Herbrand universe $U_\sigma$.

**Definition 2 (Fuzzy Interpretation)** *Let $\sigma = \langle Rel, Const, Fun \rangle$ be a signature. A fuzzy Herbrand $\sigma$-interpretation $\mathcal{I}$ over a certainty scale $\boldsymbol{C}$ consists of the canonical intrepretation of terms by themselves, and an assignment of a function $r_\mathcal{I} : U_\sigma^{a(r)} \to \boldsymbol{C}$ to every relation symbol $r \in Rel$, where $a(r)$ denotes the arity of $r$; the function $r_\mathcal{I}$ is also called a* fuzzy relation.[6]

The class of all fuzzy Herbrand $\sigma$-interpretations is denoted by $\boldsymbol{I}^+(\sigma)$. In the sequel we simply say ‘fuzzy interpretation’ instead of ‘fuzzy Herbrand interpretation’. We prefer to call the logic based on fuzzy interpretations *semi-possibilistic*,

[5] We do not question that, like probability theory, possibility theory models certain forms of reasoning with uncertainty based on the non-compositional set-algebraic treatment of events.

[6] In [VP96], a fuzzy Herbrand interpretation was defined as a function $I : \mathrm{At} \to [0, 1]$, assigning certainty values to elements of the Herbrand base. This is equivalent to our definition.

since the term 'fuzzy logic' is already widely used (with various meanings), and is connected with Zadeh's definition of negation which we find inappropriate because it leads to the violation of the *law of the excluded contradiction.*

**Definition 3 (Semi-Possibilistic Satisfaction)** [7] *Let* $F, G \in L^0(\sigma, +)$, $r(\mathbf{t}) \in \mathrm{At}^0(\sigma)$, $\mathcal{I} \in \boldsymbol{I}^+(\sigma)$, *and* $\mu, \nu \in \boldsymbol{C}$.

$$(a) \qquad \mathcal{I} \models r(\mathbf{t}){:}\mu \;:\Longleftrightarrow\; r_{\mathcal{I}}(\mathbf{t}) \geq \mu$$

$$(\neg) \qquad \mathcal{I} \models (\neg F) : \mu \;:\Longleftrightarrow\; \begin{cases} \mu > 0 \;\&\; \text{there is no } \nu > 0 \text{ s.th. } \mathcal{I} \models F{:}\nu \\ \mu = 0 \;\&\; \mathcal{I} \models F{:}\nu \text{ for some } \nu > 0 \end{cases}$$

$$(\wedge)\; \mathcal{I} \models (F \wedge G) : \mu \;:\Longleftrightarrow\; \mathcal{I} \models F{:}\mu \;\&\; \mathcal{I} \models G{:}\mu$$

$$(\vee)\; \mathcal{I} \models (F \vee G) : \mu \;:\Longleftrightarrow\; \mathcal{I} \models F{:}\mu \text{ or } \mathcal{I} \models G{:}\mu$$

**Observation 1** *If* $\mathcal{I} \models F{:}\mu$, *and* $\nu < \mu$, *then* $\mathcal{I} \models F{:}\nu$.

Proof: by straightforward induction on the complexity of $F$.

Notice that according to $(\neg)$, a negation $\neg F$ is satisfied by an interpretation $\mathcal{I}$ (with any positive certainty degree) iff $\mathcal{I}$ satisfies $F$ only with complete uncertainty. Thus, whenever a weakly negated sentence holds with some positive certainty degree, it also holds with certainty degree 1. Weak negation is, in this sense, 2-valued.

On the basis of an interpretation $\mathcal{I}$ we can assign the most informative certainty degree supported by $\mathcal{I}$ to a sentence.

**Definition 4 (Semi-Possibilistic Certainty Valuation)** *Let* $\mathcal{I} \in \boldsymbol{I}^+(\sigma)$, $r(\mathbf{t}) \in \mathrm{At}^0(\sigma)$, *and* $F, G \in L^0(\sigma)$.

$$(a) \qquad C_{\mathcal{I}}(r(\mathbf{t})) = r_{\mathcal{I}}(\mathbf{t})$$

$$(\neg) \qquad C_{\mathcal{I}}(\neg F) = \begin{cases} 1 \text{ if } C_{\mathcal{I}}(F) = 0 \\ 0 \text{ otherwise} \end{cases}$$

$$(\wedge)\; C_{\mathcal{I}}(F \wedge G) = \min(C_{\mathcal{I}}(F), C_{\mathcal{I}}(G))$$

$$(\vee)\; C_{\mathcal{I}}(F \vee G) = \max(C_{\mathcal{I}}(F), C_{\mathcal{I}}(G))$$

Semi-possibilistic certainty valuations are closely related to *necessity measures* in the sense of [DLP94]. While necessity measures, however, sacrifice compositionality in favor of 'classicality' (i.e. for preserving classical tautologies), our semi-possiblistic certainty valuations preserve compositionality on the basis of a Heyting algebra.[8]

Notice that neither the *law of the excluded middle*, $C_{\mathcal{I}}(F \vee \neg F) = 1$, nor *double negation elimination*, $C_{\mathcal{I}}(\neg\neg F) = C_{\mathcal{I}}(F)$, are valid in semi-possibilistic logic. This is completely acceptable from a logical and cognitive point of view, at least to the same degree as intuitionistic logic is acceptable. The dual of the law of the excluded middle, which is much more fundamental, does hold, however.

[7] In this paper, we define the satisfaction relation of our (semi-)possibilistic logic only for quantifier-free sentences. We will present its full definition in the extended version of the paper.

[8] $\langle [0,1], *, \min, \max \rangle$ is the implication-free fragment of the corresponding linear Heyting algebra.

**Observation 2 (Law of the Excluded Contradiction)** *For any* $\mathcal{I} \in \boldsymbol{I}^+(\sigma)$, $C_{\mathcal{I}}(F \wedge \neg F) = 0$.

*Proof.* We have to prove that either $C_{\mathcal{I}}(F)$ or $C_{\mathcal{I}}(\neg F)$ is equal to 0. This is the case, since by definition, $C_{\mathcal{I}}(\neg F) = 0$ whenever $C_{\mathcal{I}}(F) > 0$.

The following observation shows that the min/max-evaluation of conjunction and disjunction is not a matter of choice (as suggested by the t-norm approach to fuzzy logic), but is implied by the semantics of (semi-)possibilistic logic, via the clauses ($\wedge$) and ($\vee$) in the definition of the satisfaction relation.

**Claim 1** $C_{\mathcal{I}}(F)$ *assigns the most informative certainty degree supported by* $\mathcal{I}$ *to* $F$*:*

$$C_{\mathcal{I}}(F) = \max\{\mu \mid \mathcal{I} \models F{:}\mu\}$$

*or in other words,* $\mathcal{I} \models F{:}\mu$ *iff* $C_{\mathcal{I}}(F) \geq \mu$.

*Proof.* In the case of atoms and negations, the assertion follows immediately from the definitions ($a$) and ($\neg$). Let $F = G \wedge H$. Then, using observation 1,

$$\begin{aligned} C_{\mathcal{I}}(F) &= \min(C_{\mathcal{I}}(G), C_{\mathcal{I}}(H)) \\ &= \min(\max\{\mu \mid \mathcal{I} \models G{:}\mu\}, \max\{\mu \mid \mathcal{I} \models H{:}\mu\}) \\ &= \max\{\mu \mid \mathcal{I} \models G{:}\mu \ \ \& \ \ \mathcal{I} \models H{:}\mu\} \\ &= \max\{\mu \mid \mathcal{I} \models F{:}\mu\} \end{aligned}$$

Similarly for $F = G \vee H$:

$$\begin{aligned} C_{\mathcal{I}}(F) &= \max(C_{\mathcal{I}}(G), C_{\mathcal{I}}(H)) \\ &= \max(\max\{\mu \mid \mathcal{I} \models G{:}\mu\}, \max\{\mu \mid \mathcal{I} \models H{:}\mu\}) \\ &= \max\{\mu \mid \mathcal{I} \models G{:}\mu \text{ or } \mathcal{I} \models H{:}\mu\} \\ &= \max\{\mu \mid \mathcal{I} \models F{:}\mu\} \end{aligned}$$

Let $X \subseteq L^0(\sigma) \times [0,1]$ be a set of valuated sentences. The class of all models of $X$ is defined by

$$\mathrm{Mod}(X) = \{\mathcal{I} \in \boldsymbol{I}^+(\sigma) \mid \mathcal{I} \models F, \text{ for all } F \in X\}$$

and $\models$ denotes the corresponding entailment relation, i.e. $X \models F$ iff $\mathrm{Mod}(X) \subseteq \mathrm{Mod}(\{F\})$.

**Example 1** $\{p(c){:}0.7,\ q(c){:}0.2,\ q(d){:}0.4\} \models (p(c) \wedge q(c)) : 0.2$*, since every model of the premise set must assign at least 0.7 to* $p(c)$*, and 0.2 to* $q(c)$*, so it satisfies both* $q(c){:}0.2$ *and* $p(c){:}0.2$*.*

**Definition 5 (Diagram)** *The* diagram *of a fuzzy interpretation* $\mathcal{I} \in \boldsymbol{I}^+(\sigma)$ *is defined as*

$$D_{\mathcal{I}} = \{r(t_1, \ldots, t_n){:}\nu \mid \nu = r_{\mathcal{I}}(t_1, \ldots, t_n) \neq 0\}$$

Notice that in the diagram of an interpretation, uninformative sentences of the form $a{:}0$ are not included. We call a set of valuated atoms *normalized* if it contains only maximal elements (i.e. it is not the case that for any atom $a$ there are two elements $a{:}\mu$ and $a{:}\nu$ such that $\mu \neq \nu$).

**Observation 3** *Fuzzy Herbrand interpretations can be identified with their diagrams. In other words, there is a one-to-one correspondence between the class of fuzzy Herbrand interpretations and the collection of all normalized sets of valuated atoms.*

Consequently, fuzzy Herbrand interpretations over $\sigma$ can be considered as normalized subsets of $\mathrm{At}^0(\sigma) \times (0, 1]$. In the sequel, we identify an interpretation with its diagram whenever appropriate.

**Observation 4** *An ordinary Herbrand interpretation $I \subseteq \mathrm{At}^0(\sigma)$ can be embedded in a fuzzy Herbrand interpretation $I' = \{a{:}1 \mid a \in I\}$, such that for all $F \in L^0(\sigma)$,*

1. *If $I \models F$, then $C_{I'}(F) = 1$.*
2. *If $I \models \neg F$, then $C_{I'}(F) = 0$.*

*Proof.* By straightforward induction on sentences.

## 5 Possibilistic Logic

A *possibilistic Herbrand interpretation* of the language $L(\sigma)$ is based on *possibilistic relations* over the Herbrand universe $U_\sigma$. The generalization step from fuzzy to possibilistic interpretations is analogous to the generalization step from classical to partial interpretations. Recall that a general partial interpretation assigns falsity/truth-values from $\{0, 1\} \times \{0, 1\}$, where $\langle 1, 0\rangle$ denotes *false*, $\langle 0, 1\rangle$ denotes *true*, $\langle 0, 0\rangle$ denotes *undetermined*, and $\langle 1, 1\rangle$ denotes *overdetermined*. Classical logic can be obtained from partial logic by admitting only total coherent models, see [HJW96].

**Definition 6 (General Possibilistic Interpretation)** *A general possibilistic Herbrand $\sigma$-interpretation $\mathcal{I}$ over a certainty scale $\boldsymbol{C}$ consists of the Herbrand universe $U_\sigma$, the canonical interpretation of constants, function symbols and terms, and an assignment of a function $r_\mathcal{I} : U_\sigma^{a(r)} \to \boldsymbol{C} \times \boldsymbol{C}$ to every relation symbol $r \in Rel$, where $a(r)$ denotes the arity of $r$; the function $r_\mathcal{I}$ is called a* possibilistic relation, *and we also write*

$$r_\mathcal{I}(\mathbf{t}) = \langle r_\mathcal{I}^-(\mathbf{t}), r_\mathcal{I}^+(\mathbf{t})\rangle$$

*$\mathcal{I}$ is called*

1. coherent, *if for no tuple $\mathbf{t}$, $r_\mathcal{I}^+(\mathbf{t}) > 0$ & $r_\mathcal{I}^-(\mathbf{t}) > 0$.*
2. total, *if for no tuple $\mathbf{t}$, $r_\mathcal{I}^+(\mathbf{t}) = r_\mathcal{I}^-(\mathbf{t}) = 0$.*

Obviously, fuzzy interpretations can be viewed as particular possibilistic interpretations not assigning certainty degrees of falsity but only of truth: a fuzzy relation $r_{\mathcal{I}}$ corresponds to the possibilistic relation $\widehat{r_{\mathcal{I}}} : U_\sigma^{a(r)} \to \{0\} \times \boldsymbol{C}$ with $\widehat{r_{\mathcal{I}}}(\mathbf{t}) = \langle 0, r_{\mathcal{I}}(\mathbf{t})\rangle$.

In the framework of [DLP94], the functions $r^+_{\mathcal{I}}$ and $r^-_{\mathcal{I}}$ correspond to *necessity* and *impossibility* measures.[9] However, Dubois, Prade and Lang do not treat two kinds of negation which, in our view, is essential for possibilistic logic and possibilistic logic programs to the same extent as it is for partial logic and extended logic programs. In the sequel, we will only consider coherent possibilistic interpretations, and we simply say 'possibilistic interpretation' or 'interpretation' instead of 'coherent possibilistic Herbrand interpretation' whenever the context allows it. The class of all coherent possibilistic Herbrand $\sigma$-interpretations is denoted by $\boldsymbol{I}(\sigma)$.

Formally, a possibilistic certainty valuation assigns a pair of certainty degrees $\langle r^-_{\mathcal{I}}(\mathbf{t}), r^+_{\mathcal{I}}(\mathbf{t})\rangle$ to an atomic sentence $r(\mathbf{t})$, such that the first value denotes the certainty of falsity, and the second value the certainty of truth. The certainty value of a strongly negated atom $\sim r(\mathbf{t})$, then, is obtained by swapping the truth and falsity degree, i.e. it is equal to $\langle r^+_{\mathcal{I}}(\mathbf{t}), r^-_{\mathcal{I}}(\mathbf{t})\rangle$. Since we only deal with coherent possibilistic interpretations, we can simplify the notation using a *signed* certainty scale consisting of a positive and a negative range: $\pm\boldsymbol{C} = \boldsymbol{C} \cup \{-v : v \in \boldsymbol{C}\}$. We also write $[-1, 1]$ instead of $\pm\boldsymbol{C}$. The information ordering $\leq_\pm$ of $\pm\boldsymbol{C}$ is defined as: $\mu \leq_\pm \nu$ iff $\mu, \nu < 0$ & $\nu \leq \mu$, or $\mu, \nu > 0$ & $\mu \leq \nu$. Notice that 0 stands for complete uncertainty, or *no information*, whereas $-1$ stands for *completely certain falsity*, or impossibility.

**Definition 7 (Possibilistic Satisfaction)** *Let* $F, G \in L^0(\sigma)$, $r(\mathbf{t}) \in \mathrm{At}^0(\sigma)$, $\mathcal{I} \in \boldsymbol{I}(\sigma)$, *and* $\mu \in [-1, 1]$.

$$
\begin{array}{ll}
(a) & \mathcal{I} \models r(\mathbf{t}){:}\mu \;:\Longleftrightarrow \begin{cases} \mu \geq 0 \;\&\; r^+_{\mathcal{I}}(\mathbf{t}) \geq \mu \\ \mu < 0 \;\&\; r^-_{\mathcal{I}}(\mathbf{t}) \geq -\mu \end{cases} \\
(\neg) & \mathcal{I} \models (\neg F){:}\mu \;:\Longleftrightarrow \begin{cases} \mu \geq 0 \;\&\; \textit{there is no } \nu > 0 \textit{ s.th. } \mathcal{I} \models F{:}\nu \\ \mu \leq 0 \;\&\; \mathcal{I} \models F{:}\nu \textit{ for some } \nu > 0 \end{cases} \\
(\sim) & \mathcal{I} \models (\sim F){:}\mu \;:\Longleftrightarrow \mathcal{I} \models F{:}-\mu \\
(\wedge) & \mathcal{I} \models (F \wedge G) : \mu \;:\Longleftrightarrow \begin{cases} \mu \geq 0 \;\&\; (\mathcal{I} \models F{:}\mu \;\&\; \mathcal{I} \models G{:}\mu) \\ \mu < 0 \;\&\; (\mathcal{I} \models F{:}\mu \textit{ or } \mathcal{I} \models G{:}\mu) \end{cases} \\
(\vee) & \mathcal{I} \models (F \vee G) : \mu \;:\Longleftrightarrow \begin{cases} \mu \geq 0 \;\&\; (\mathcal{I} \models F{:}\mu \textit{ or } \mathcal{I} \models G{:}\mu) \\ \mu < 0 \;\&\; (\mathcal{I} \models F{:}\mu \;\&\; \mathcal{I} \models G{:}\mu) \end{cases}
\end{array}
$$

**Observation 5** *If* $\mathcal{I} \models F{:}\mu$, *and* $\nu <_\pm \mu$, *then* $\mathcal{I} \models F{:}\nu$.

**Observation 6 (Coherence Principle)**

1. *If* $\mathcal{I} \models \sim F : \mu$ *and* $\mu > 0$, *then* $\mathcal{I} \models \neg F : 1$. *In other words,* $C_{\mathcal{I}}(\sim F) > 0$, *or equivalently* $C_{\mathcal{I}}(F) < 0$, *implies* $C_{\mathcal{I}}(\neg F) = 1$.

[9] An impossibility measure is simply the inverse $1 - \pi$ of a possibilty measure $\pi$.

2. *If $\mathcal{I} \models \sim F : \mu$ and $\mu < 0$, then $\mathcal{I} \models \neg F : -1$. In other words, $C_\mathcal{I}(\sim F) < 0$, or equivalently $C_\mathcal{I}(F) > 0$, implies $C_\mathcal{I}(\neg F) = -1$.*

**Definition 8 (Possibilistic Certainty Valuation)**

$$\begin{array}{ll} (a) & C_\mathcal{I}(r(\mathbf{t})) = \begin{cases} r^+_\mathcal{I}(\mathbf{t}) & \textit{if } r^-_\mathcal{I}(\mathbf{t}) = 0 \\ -r^-_\mathcal{I}(\mathbf{t}) & \textit{otherwise} \end{cases} \\ (\neg) & C_\mathcal{I}(\neg F) = \begin{cases} 1 \textit{ if } C_\mathcal{I}(F) \leq 0 \\ -1 \textit{ otherwise} \end{cases} \\ (\sim) & C_\mathcal{I}(\sim F) = -C_\mathcal{I}(F) \\ (\wedge) & C_\mathcal{I}(F \wedge G) = \min(C_\mathcal{I}(F), C_\mathcal{I}(G)) \\ (\vee) & C_\mathcal{I}(F \vee G) = \max(C_\mathcal{I}(F), C_\mathcal{I}(G)) \end{array}$$

Notice that 'min' and 'max' in $(\wedge)$ and $(\vee)$ refer to $\leq$ and not to $\leq_\pm$.

**Claim 2** *$C_\mathcal{I}(F)$ assigns the most informative certainty degree supported by $\mathcal{I}$ to $F$:*

$$C_\mathcal{I}(F) = \max_\pm \{\mu \mid \mathcal{I} \models F{:}\mu\}$$

*or in other words, $\mathcal{I} \models F{:}\mu$ iff $C_\mathcal{I}(F) \geq_\pm \mu$.*

**Observation 7** *For any $\mathcal{I} \in \boldsymbol{I}(\sigma)$, $C_\mathcal{I}(F \wedge \neg F) = C_\mathcal{I}(F \wedge \sim F) = 0$.*

The model operator Mod and the entailment relation $\models$ are defined in the standard way (see previous section). The *diagram* of a possibilistic interpretation $\mathcal{I} \in \boldsymbol{I}(\sigma)$ is defined as

$$D_\mathcal{I} = \{r(\mathbf{t}){:}\nu \mid \nu = r_\mathcal{I}(\mathbf{t}) \neq 0\}$$

Notice that in the diagram of a possibilistic interpretation, there are only informative sentences of the form $a{:}c$ or $a{:}{-c}$, where $c \in (0,1]$. We identify possibilistic Herbrand interpretations with their diagrams, i.e. possibilistic Herbrand interpretations over $\sigma$ are normalized subsets of $\mathrm{At}^0(\sigma) \times [-1,0) \cup (0,1]$, or equivalently of $\mathrm{Cons}(\mathrm{Lit}^0(\sigma)) \times (0,1]$.

**Claim 3** *A coherent partial Herbrand interpretation $I \subseteq \mathrm{Lit}^0(\sigma)$ can be embedded in a possibilistic Herbrand interpretation $I' = \{a{:}1 \mid a \in I\} \cup \{a : -1 \mid \sim a \in I\}$, such that for all $F \in L^0(\sigma)$,*

1. *If $I \models F$, then $C_{I'}(F) = 1$.*
2. *If $I \models \neg F$, then $C_{I'}(F) = 0$, or $C_{I'}(F) = -1$.*
3. *If $I \models \sim F$, then $C_{I'}(F) = -1$.*

## 6 Databases and Minimal Models

**Definition 9 (Fuzzy and Possibilistic Databases)** *A fuzzy (resp. possibilistic) database is a finite set of finite fuzzy (resp. possibilistic) relations (or 'tables'), corresponding to a normalized set of certainty-valuated atoms (resp. literals).*

Thus, a database corresponds to a finite interpretation.

**Example 2** *The possibilistic database consisting of the two tables*

$$P = \begin{array}{|c|c|} \hline d & -1 \\ b & 0.7 \\ c & 0.1 \\ \hline \end{array} \qquad Q = \begin{array}{|c|c|} \hline d\ c & 0.8 \\ d\ b & -0.3 \\ \hline \end{array}$$

*corresponds to* $X_2 = \{p(d){:}-1,\ p(b){:}0.7,\ p(c){:}0.1,\ q(d,c){:}0.8,\ q(d,b){:}-0.3\}$.

**Definition 10 (Informational Extension)** *Let* $\mathcal{I}, \mathcal{I}' \in \boldsymbol{I}$ *be two interpretations. We say that* $\mathcal{I}'$ informationally extends $\mathcal{I}$, *or* $\mathcal{I}'$ *is* at least as informative *as* $\mathcal{I}$, *symbolically* $\mathcal{I} \leq \mathcal{I}'$, *if for all* $a{:}\nu \in D_{\mathcal{I}}$ *exists* $a{:}\mu \in D_{\mathcal{I}'}$, *such that* $\nu \leq_{\pm} \mu$.

This means that an interpretation (or a table) contains more information than another one, if it contains additional elements (entries), or the certainty of some elements in it is increased. For instance,

$$\begin{array}{|c|c|} \hline d & -0.9 \\ b & 0.7 \\ \hline \end{array} \quad < \quad \begin{array}{|c|c|} \hline d & -0.9 \\ b & 0.7 \\ c & 0.1 \\ \hline \end{array} \quad < \quad \begin{array}{|c|c|} \hline d & -1 \\ b & 0.7 \\ c & 0.1 \\ \hline \end{array}$$

**Definition 11 (Minimal Model)** *For* $F \in L^0(\sigma) \times [-1,1] \supseteq X$, *we define* $\mathrm{Mod}_m(X) = \mathrm{Min}(\mathrm{Mod}(X))$, *and minimal entailment:* $X \models_m F$ *iff* $\mathrm{Mod}_m(X) \subseteq \mathrm{Mod}(F)$.

**Definition 12 (Natural Inference)** *A valuated sentence* $F$ *can be inferred from a fuzzy (or possibilistic) database* $X$ *if it is minimally entailed:* $X \vdash F$ *if* $X \models_m F$.

For instance, $X_2 \vdash \neg q(b,c) : 1 \wedge {\sim} p(d) : 1$.

## 7 Related Work

The main problem for many fuzzy logic programming approaches, such as [vEm86, EM95, VP96] is the fuzzy logic evaluation of negation: $C(\neg F) = 1 - C(F)$, which is neither justified by any reasonable logical semantics nor compatible with the treatment of negation in logic programs. As a consequence of this problem, neither of these approaches allows for negation in the body of a rule. We have remedied the negation problem of fuzzy logic by proposing our *semi-possibilistic* logic where we combine the min/max-evaluation of conjunction and disjunction with a supraintuitionistic evaluation of weak negation, thereby preserving the law of the excluded contradiction and providing a semantic link to negation-as-failure. By defining a satisfaction relation for uncertainty-qualified formulas, we show that the evaluation of conjunction and disjunction is not a matter of choice, as suggested by the t-norm approach to fuzzy logic (adopted, e.g., in [EM95, VP96]), but has to be done by means of minimum and maximum.

In [DLP91], the authors present a possibilistic extension of positive logic programs. Since negation is not allowed, rules cannot have negative conditions. The negation in the propositional possibilistic logic of [DLP94] corresponds rather to our strong negation $\sim$, and not to the weak negation $\neg$ used in normal logic programs (as 'negation by failure'). Thus, normal logic programs cannot be embedded.

## 8 Conclusion

We have shown how to combine the notion of uncertain infomation with the semantical framework of logic programming, i.e. with minimal and stable Herbrand models. Our notion of uncertainty is derived from the possibility theory of Zadeh, Dubois and Prade which seems to be the natural choice in the context of an information-based semantics (as opposed to a truth-based semantics where probability theory may be more appropriate).

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