A Logical Reconstruction of Fuzzy Inference in Databases and Logic Programs

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Abstract

We propose to replace Zadeh's DeMorgan-type negation in fuzzy logic by a Heyting-type negation which, unlike the former, preserves the law of the excluded contradiction and is more in line with negation in databases and logic programs. We show that the resulting system can be used for obtaining conservative extensions of relational and deductive databases (resp. normal logic programs).

1 Introduction

The semantics of databases and logic programs is defined on the basis of intended Herbrand models. We show that this also holds for fuzzy databases and logic programs. There are, however, two possibilities how to evaluate negation in fuzzy Herbrand interpretations based on linear certainty scales. As an alternative to Zadeh's DeMorgan-type negation, we propose a Heyting-type negation which preserves the law of the excluded contradiction, and corresponds to 'negation-as-failure' in fuzzy logic programs. We call the resulting system *semi-possibilistic logic*.

In the following heuristic deduction rule about the diagnosis of hepatitis the absence of positive evidence is required by the negated conditions (similar to negation-as-failure):

diagnosis(Patient, hepatitis_X)
⟵ diagnostic_finding(Patient, cirrhosis),
¬ diagnosis(Patient, hepatitis_A),
¬ diagnosis(Patient, hepatitis_B),
...
¬ diagnosis(Patient, hepatitis_G).

This rule expresses the heuristic, that if there is evidence for the diagnostic finding *cirrhosis* (say, its certainty is 0.8), and there is no evidence for a diagnosis of either of hepatitis A to G, then the diagnosis is hepatitis X (with certainty 0.8). If there is any evidence for either of hepatitis A to G, the rule should fail (i.e. not produce any evidence in favor of hepatitis X). This will not be the case, however, if ¬ denotes the ordinary negation of fuzzy logic, where the negation of a sentence F is evaluated with the positive certainty value $1 - C(F)$ whenever F itself has a certainty value $C(F) < 1$.

In real world knowledge bases like, for instance, relational or deductive databases, it is essential to be able to infer negative information by means of *minimal* (or *stable*) entailment, i.e. drawing inferences on the basis of minimal (or stable) models. It turns out that for fuzzy logic programs, like for normal logic programs, minimal models are not adequate because they are not able to account for the directedness of rules. Therefore, the more refined preference criterion of stability is needed to capture the class of intended models. In this paper, we introduce the new notion of *stable generated models* of fuzzy logic programs, thereby laying the foundations of nonmonotonic reasoning with fuzzy information based on the preferential semantics of stable generated models. Our stable semantics of fuzzy logic programs is not restricted to any specific rule format. In fact, it admits of rules with or without weights, and it allows for arbitrary formulas in both the body and the head of a rule, including the case of disjunction and negation in the head.

In the literature, there is no clear taxonomy of uncertainty in databases and logic programs. An obvious distinction, however, concerns the uncertainty expressed at the level of attribute values, or at the level of the applicability of a predicate to a tuple (resp. its membership in a class of objects). While the former has been associated with the issue of handling *vagueness*, the latter corresponds to an account of uncertainty-qualified sentences. In this paper we do not treat vagueness due to fuzzy-set-valued attributes, but only the more fundamental issue of gradual uncertainty based on fuzzy relations over ordinary (i.e. crisp) attributes.

The structure of the paper is as follows. In section 3, we define *semi-possibilistic* logic. In section 4, we define a natural information ordering between fuzzy interpretations which is the basis for the notions of minimal and stable models. Fuzzy logic programs and stable models are defined in section 5, where we show that normal logic programs under the stable

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model semantics of [5] can be faithfully embedded in fuzzy logic programs under our stable semantics.

2 Preliminaries

A predicate logic *signature* $\sigma = \langle Rel, Const, Fun \rangle$ consists of a set of relation symbols, a set of constant symbols, and a set of function symbols. U_σ denotes the set of all ground terms of σ. For a term tuple $t_1, \ldots, t_n$ we also write $\boldsymbol{t}$ when its length is of no relevance. The logical functors are $\neg, \wedge, \vee$. $L(\sigma)$ is the smallest set containing the atomic formulas of σ, and being closed with respect to the following conditions: if $F, G \in L(\sigma)$, then $\{\neg F, F \wedge G, F \vee G\} \subseteq L(\sigma)$. $\mathrm{At}(\sigma)$ denotes the set of all atomic formulas (also called *atoms*). When $L \subseteq L(\sigma)$ is some sublanguage, L^0 denotes the corresponding set of sentences (closed formulas). If the signature σ does not matter, we omit it and write, e.g., L instead of $L(\sigma)$.

If Y is a preorder, then $\mathrm{Min}(Y)$ denotes the set of all minimal elements of Y, i.e. $\mathrm{Min}(Y) = \{X \in Y \mid \neg\exists X' \in Y : X' < X\}$.

3 Semi-Possibilistic Logic

Definition 1 (Certainty Scale) *A certainty scale* $\langle \boldsymbol{C}, 0, 1 \rangle$ *is a linearly ordered set* $\boldsymbol{C}$ *with least and greatest elements* 0 *and* 1.

Examples of certainty scales are the rational unit interval, or any discrete ordering of linguistic uncertainty values such as $\langle 0, ll, ql, vl, 1 \rangle$, where ll stands for *little likely*, ql for *quite likely*, and vl for *very likely*. In the sequel, we assume that there is a fixed certainty scale $\boldsymbol{C}$ for which we simply write $[0, 1]$. If we want to exclude the value for complete uncertainty, we write $(0, 1] = \{v \in \boldsymbol{C} \mid v > 0\}$.

It is important to note that in semi-possibilistic logic, unlike in probability theory, the intuitive meaning of 0 is not *false*, or impossible, but rather *completely uncertain*, or absolutely *no evidence*.

Observation 1 *A certainty scale* $\boldsymbol{C}$ *corresponds to a* Heyting algebra *whose implication-free fragment is* $\langle \boldsymbol{C}, \min, \max, \frown \rangle$, *where the Heyting complement* $\frown$ *is defined as*

$$\frown v = \begin{cases} 0 & \text{if } v > 0 \\ 1 & \text{otherwise} \end{cases}$$

A *fuzzy Herbrand interpretation* of the language $L(\sigma)$ is based on fuzzy relations over the Herbrand universe U_σ.

Definition 2 (Fuzzy Interpretation) *Let* $\sigma = \langle Rel, Const, Fun \rangle$ *be a signature. A fuzzy Herbrand* σ*-interpretation* $\mathcal{I}$ *over a certainty scale* $\boldsymbol{C}$ *consists of the canonical intrepretation of terms by themselves, and an assignment of a function* $r_\mathcal{I} : U_\sigma^{a(r)} \to \boldsymbol{C}$ *to every relation symbol* $r \in Rel$, *where* $a(r)$ *denotes the arity of* r*; the function* $r_\mathcal{I}$ *is also called a* fuzzy relation.[1]

The class of all fuzzy Herbrand σ-interpretations is denoted by $\boldsymbol{I}^f(\sigma)$. In the sequel we simply say 'fuzzy interpretation' instead of 'fuzzy Herbrand interpretation'. We prefer to call the logic based on fuzzy interpretations *semi-possibilistic*, since the term 'fuzzy logic' is already widely used (with various meanings), and is connected with Zadeh's definition of negation, i.e. $C(\neg F) = 1 - C(F)$, which we find inappropriate because it leads to the violation of the *law of the excluded contradiction*, i.e. it does not hold that $C(F \wedge \neg F) = 0$.

Notice that, strictly speaking, our semi-possibilistic logic is not a multi-valued logic. Since it is defined for arbitrary certainty scales, it rather corresponds to the supraintuitionistic logic generated by linear orderings which is also called *Dummett's Logic*.

The semi-possibilistic satisfaction relation between a fuzzy Herbrand interpretation and a certainty-valuated sentence is defined in figure 1. For brevity, we define the satisfaction relation here only for quantifier-free sentences. We use $\sim$ to denote the DeMorgan-type negation proposed by Zadeh, and $\neg$ to denote the Heyting-type negation which we propose as the better choice.

Observation 2 *If* $\mathcal{I} \models F{:}\mu$, *and* $\nu < \mu$, *then* $\mathcal{I} \models F{:}\nu$.

Notice that according to $(\neg)$, a negation $\neg F$ is satisfied by an interpretation $\mathcal{I}$ (with any positive certainty degree) iff $\mathcal{I}$ is completely uncertain about F. Thus, whenever a negated sentence holds with some positive certainty degree, it also holds with absolute certainty ($\neg$ is, in this sense, 2-valued).

An interpretation $\mathcal{I}$ induces a certainty valuation.

Definition 3 (Certainty Valuation) *Let* $\mathcal{I} \in \boldsymbol{I}^f(\sigma)$, $r(\boldsymbol{t}) \in \mathrm{At}^0(\sigma)$, *and* $F, G \in L^0(\sigma)$.

$$\begin{array}{lrcl}
(a) & C_\mathcal{I}(r(\boldsymbol{t})) & = & r_\mathcal{I}(\boldsymbol{t}) \\
(\sim) & C_\mathcal{I}(\sim F) & = & 1 - C_\mathcal{I}(F) \\
(\neg) & C_\mathcal{I}(\neg F) & = & \frown C_\mathcal{I}(F) \\
(\wedge) & C_\mathcal{I}(F \wedge G) & = & \min(C_\mathcal{I}(F), C_\mathcal{I}(G)) \\
(\vee) & C_\mathcal{I}(F \vee G) & = & \max(C_\mathcal{I}(F), C_\mathcal{I}(G))
\end{array}$$

Semi-possibilistic certainty valuations are closely related to *necessity measures* in the sense of [2]. While

[1] In [9], a fuzzy Herbrand interpretation was defined as a function $I : \mathrm{At} \to [0, 1]$, assigning certainty values to elements of the Herbrand base. This is equivalent to our definition.

Let $F, G \in L^0(\sigma, \neg, \wedge, \vee)$, $r(\boldsymbol{t}) \in \mathrm{At}^0(\sigma)$, $\mathcal{I} \in \boldsymbol{I}^f(\sigma)$, and $\mu, \nu \in \boldsymbol{C}$.

$$
\begin{array}{lrcl}
(a) & \mathcal{I} \models r(\boldsymbol{t}){:}\mu & :\Longleftrightarrow & r_{\mathcal{I}}(\boldsymbol{t}) \geq \mu \\
(\sim) & \mathcal{I} \models ({\sim}F) : \mu & :\Longleftrightarrow & \text{there is no } \nu > 1-\mu \text{ s.th. } \mathcal{I} \models F{:}\nu \\
(\neg) & \mathcal{I} \models (\neg F) : \mu & :\Longleftrightarrow & \begin{cases} \mu > 0 \ \& \ \text{ there is no } \nu > 0 \text{ s.th. } \mathcal{I} \models F{:}\nu \\ \mu = 0 \ \& \ \mathcal{I} \models F{:}\nu \text{ for some } \nu > 0 \end{cases} \\
(\wedge) & \mathcal{I} \models (F \wedge G) : \mu & :\Longleftrightarrow & \mathcal{I} \models F{:}\mu \ \& \ \mathcal{I} \models G{:}\mu \\
(\vee) & \mathcal{I} \models (F \vee G) : \mu & :\Longleftrightarrow & \mathcal{I} \models F{:}\mu \text{ or } \mathcal{I} \models G{:}\mu
\end{array}
$$

Figure 1: Semi-Possibilistic Satisfaction

necessity measures, however, sacrifice compositionality in favor of 'classicality' (i.e. for preserving classical tautologies),[2] our semi-possiblistic certainty valuations preserve compositionality on the basis of a Heyting algebra which seems to be the better choice since compositionality is more fundamental than classical tautologies.

Notice that while the *Law of the Excluded Middle (LEM)* is violated by both negations: for certain sentences F, neither $C_{\mathcal{I}}(F \vee \neg F) = 1$, nor $C_{\mathcal{I}}(F \vee {\sim}F) = 1$,[3] its dual, the *Law of the Excluded Contradiction (LEC)* which is much more fundamental, does hold for $\neg$ but not for $\sim$.

Observation 3 (LEC) $C_{\mathcal{I}}(F \wedge \neg F) = 0$.

Proof: We have to prove that either $C_{\mathcal{I}}(F)$ or $C_{\mathcal{I}}(\neg F)$ is equal to 0. This is the case, since by definition, $C_{\mathcal{I}}(\neg F) = 0$ whenever $C_{\mathcal{I}}(F) > 0$. □

The law of *double negation elimination*, $C_{\mathcal{I}}({\sim}{\sim}F) = C_{\mathcal{I}}(F)$, does hold for the DeMorgan-type negation $\sim$, but not for the Heyting-type negation $\neg$. This is completely acceptable from a logical and cognitive point of view, at least to the same degree as intuitionistic logic is acceptable. Our considerations show that $\neg$ behaves much more in a logical way (and is closer to negation in classical logic) than $\sim$, since it satisfies (LEC) and a weak form of (LEM).

The following claim shows that the min/max-evaluation of conjunction and disjunction is not a matter of choice (as suggested by the t-norm approach to fuzzy logic), but is implied by the semantics of semi-possibilistic logic, via the clauses $(\wedge)$ and $(\vee)$ in the definition of the satisfaction relation.

Claim 1 *$C_{\mathcal{I}}(F)$ assigns the most informative certainty degree supported by $\mathcal{I}$ to F:*

$$C_{\mathcal{I}}(F) = \max\{\mu \mid \mathcal{I} \models F{:}\mu\}$$

or in other words, $\mathcal{I} \models F{:}\mu$ iff $C_{\mathcal{I}}(F) \geq \mu$.

Proof by induction on F: In the case of atoms and negations, the assertion follows immediately from the definitions (a), $(\sim)$ and $(\neg)$. Let $F = G \wedge H$. Then, using observation 2,

$$
\begin{aligned}
& C_{\mathcal{I}}(F) = \min(C_{\mathcal{I}}(G), C_{\mathcal{I}}(H)) \\
&= \min(\max\{\mu \mid \mathcal{I} \models G{:}\mu\}, \max\{\mu \mid \mathcal{I} \models H{:}\mu\}) \\
&= \max\{\mu \mid \mathcal{I} \models G{:}\mu \ \& \ \mathcal{I} \models H{:}\mu\} \\
&= \max\{\mu \mid \mathcal{I} \models F{:}\mu\}
\end{aligned}
$$

Similarly for $F = G \vee H$:

$$
\begin{aligned}
& C_{\mathcal{I}}(F) = \max(C_{\mathcal{I}}(G), C_{\mathcal{I}}(H)) \\
&= \max(\max\{\mu \mid \mathcal{I} \models G{:}\mu\}, \max\{\mu \mid \mathcal{I} \models H{:}\mu\}) \\
&= \max\{\mu \mid \mathcal{I} \models G{:}\mu \text{ or } \mathcal{I} \models H{:}\mu\} \\
&= \max\{\mu \mid \mathcal{I} \models F{:}\mu\} \qquad \square
\end{aligned}
$$

Let $X \subseteq L^0(\sigma) \times [0,1]$ be a set of valuated sentences. The class of all models of X is defined by

$$\mathrm{Mod}(X) = \{\mathcal{I} \in \boldsymbol{I}^f(\sigma) \mid \mathcal{I} \models F, \text{ for all } F \in X\}$$

and $\models$ denotes the corresponding entailment relation, i.e. $X \models F$ iff $\mathrm{Mod}(X) \subseteq \mathrm{Mod}(\{F\})$.

Definition 4 (Diagram) *The* diagram *of a fuzzy interpretation $\mathcal{I} \in \boldsymbol{I}^f(\sigma)$ is defined as*

$$D_{\mathcal{I}} = \{r(t_1, \ldots, t_n){:}\nu \mid \nu = r_{\mathcal{I}}(t_1, \ldots, t_n) \neq 0\}$$

Notice that in the diagram of an interpretation, uninformative sentences of the form $a{:}0$ are not included. We call a set of valuated atoms *normalized* if it contains only maximal elements (i.e. it is not the case that for any atom a there are two elements $a{:}\mu$ and $a{:}\nu$ such that $\mu \neq \nu$).

Observation 4 *Fuzzy Herbrand interpretations can be identified with their diagrams. In other words, there is a one-to-one correspondence between the class of fuzzy Herbrand interpretations and the collection of all normalized sets of valuated atoms.*

Consequently, fuzzy Herbrand interpretations over σ can be considered as normalized subsets of $\mathrm{At}^0(\sigma) \times (0,1]$. In the sequel, we identify an interpretation $\mathcal{I}$ with its diagram whenever appropriate, and write also I instead of $\mathcal{I}$.

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[2] Mainly by defining disjunction through the inequality $C_{\mathcal{I}}(F \vee G) \geq \max(C_{\mathcal{I}}(F), C_{\mathcal{I}}(G))$ instead of the above equality $(\vee)$.

[3] A weak version of the law of the excluded middle holds, however, for $\neg$: $C_{\mathcal{I}}(\neg F \vee \neg\neg F) = 1$.

Observation 5 *An ordinary Herbrand interpretation $I \subseteq \mathrm{At}^0(\sigma)$ can be embedded in a fuzzy Herbrand interpretation $I^f = \{a{:}1 \mid a \in I\}$, such that for all $F \in L^0(\sigma)$,*

1. *If $I \models F$, then $C_{I^f}(F) = 1$.*
2. *If $I \models \neg F$, then $C_{I^f}(F) = 0$, and consequently, $C_{I^f}(\neg F) = C_{I^f}(\sim F) = 1$.*

Proof: By straightforward induction on sentences. □

4 Fuzzy Databases and Minimal Models

Definition 5 (Fuzzy Databases) *A fuzzy database is a finite set of finite fuzzy relations (or* fuzzy tables*), corresponding to a normalized set of certainty-valuated atoms.*

Thus, a database X corresponds to a finite interpretation M_X. Instead of C_{M_X}, we simply write C_X for the certainty valuation induced by M_X.

Example 1 *The fuzzy database consisting of the two tables*

$P =$

d	1
b	0.7
c	0.1

$Q =$

d	c	0.8
d	b	0.3

corresponds to $X_1 = \{p(d){:}1,\ p(b){:}0.7,\ p(c){:}0.1,\ q(d,c){:}0.8,\ q(d,b){:}0.3\}$.

Definition 6 (Informational Extension) *Let I, J be two interpretations. We say that J* informationally extends *I, or J is* at least as informative *as I, symbolically $I \leq J$, if for all $a{:}\nu \in I$ exists $a{:}\mu \in J$, such that $\nu \leq \mu$.*

This means that an interpretation (or a table) contains more information than another one, if it contains additional elements (entries), or the certainty of some elements in it is increased. For instance,

d	0.9
b	0.7

$<$

d	0.9
b	0.7
c	0.1

$<$

d	1
b	0.7
c	0.1

Definition 7 (Minimal Model) *Let F be a valuated sentence, and X a set of valuated sentences. We define the minimal models of X by $\mathrm{Mod}_m(X) := \mathrm{Min}(\mathrm{Mod}(X))$, and minimal entailment: $X \models_m F$ iff $\mathrm{Mod}_m(X) \subseteq \mathrm{Mod}(\{F\})$.*

Definition 8 (Natural Inference) *Let X be a fuzzy database, and $F \in L^0$ an if-query. Then,*

$$X \vdash F{:}\mu \quad \textit{iff} \quad C_X(F) \geq \mu$$

For instance, $X_1 \vdash (\neg q(b,c) \wedge p(d)) : 1$.

Observation 6 *A valuated sentence $F{:}\mu$ can be inferred from a fuzzy database X if it is minimally entailed: $X \vdash F{:}\mu$ iff $X \models_m F{:}\mu$.*

5 Fuzzy Logic Programs and Stable Models

We allow for three kinds of rules in fuzzy logic programs.

Definition 9 (Fuzzy Deduction Rules) *Let $F, G \in L^0$ be arbitrary sentences. A fuzzy deduction rule r is an expression of the form*

(1) $F \leftarrow G$, *or*

(2) $F \overset{\mu}{\longleftarrow} G$, *where $\mu \in (0,1]$ is a rational number, or*

(3) $F{:}\mu \leftarrow G_1{:}\nu_1 \wedge \ldots \wedge G_n{:}\nu_n$.

The body *of r, denoted by Br, is given by G, resp. $G_1{:}\nu_1 \wedge \ldots \wedge G_n{:}\nu_n$, and the* head *of r, denoted by Hr, is given by F, resp. $F{:}\mu$.*[4] *In rules of the form (3), the body may be empty ($n = 0$) in which case it is trivially satisfied by every interpretation. We also write simply $F{:}\mu$ instead of $F{:}\mu \leftarrow$ for such* fuzzy facts.

A fuzzy logic program is a set of fuzzy deduction rules, or equivalently a pair $\langle X, R\rangle$, consisting of a fuzzy database (or a set of fuzzy facts) X and a set of fuzzy deduction rules R. We denote the restriction of R to rules of the form (i) by $R^{(i)}$. For simplicity, we assume that a program is fully instantiated, i.e. that all rules are variable-free (since it is clear how to instantiate a program, we will present program rules with variables).

Definition 10 (Model of a Rule) *Let I be a fuzzy Herbrand interpretation. Then,*

(1)	$I \models F \leftarrow G$ *iff* $C_I(F) \geq C_I(G)$
(2)	$I \models F \overset{\mu}{\leftarrow} G$ *iff* $C_I(F) \geq \mu C_I(G)$
(3)	$I \models r$ *iff* $I \models Br$ *implies* $I \models Hr$

where r is a rule of the form (3).

Obviously, $I \models F \overset{1}{\leftarrow} G$ iff $I \models F \leftarrow G$.

Let $\boldsymbol{K}$ be a class of interpretations. We write $\boldsymbol{K} \models F$ iff $I \models F$ for all $I \in \boldsymbol{K}$. We define the set of all rules from a set R which are applicable in $\boldsymbol{K}$ by

$$R_{\boldsymbol{K}} = \{r \in R^{(3)} \mid \boldsymbol{K} \models Br\} \quad \cup \\ \{r \in R^{(1)} \cup R^{(2)} \mid C_I(Br) > 0 \text{ for all } I \in \boldsymbol{K}\}$$

If $\boldsymbol{K}$ is a singleton, we omit brackets.

[4]Instead of the conjunction symbol $\wedge$, one also simply uses commas in the body of a rule.

5.1 Stable Models

Fuzzy logic programs may have minimal models which are not intended. This is illustrated by the following example.

Example 2
Let $R_2 = \{p(c){:}vl,\ r(x) \leftarrow p(x) \wedge \neg q(x)\}$ *be a fuzzy logic program over the discrete certainty scale* $[0,1] = \langle 0, ll, ql, vl, 1\rangle$. *The minimal models of* R_2 *are*

$$\begin{array}{rcl} M_1 & = & \{p(c){:}vl,\ r(c){:}vl\} \\ M_2 & = & \{p(c){:}vl,\ q(c){:}ll\} \end{array}$$

M_2 *is not an intended model: the program* $\Pi 2$ *does not provide any reason to believe* $q(c){:}ll$*, since this fact does not occur in any conclusion of a program rule. Only* M_1 *is an intended model, and thus* $r(c){:}vl$ *should be inferrable.*

Therefore, instead of minimality we need a more refined preference criterion which allows to select the intended models of a program from its Herbrand models.

Definition 11 (Interpretation Interval)
$[M_1, M_2] = \{M \in \boldsymbol{I} : M_1 \leq M \leq M_2\}$

The following definition of a *stable generated model* of a fuzzy logic program was introduced for normal logic programs in [6].

Definition 12 (Stable Generated Model)
A model M *of a fuzzy logic program* R *is called* stable generated, *symbolically* $M \in \mathrm{Mod}_s(R)$*, if there is a chain of interpretations* $I_0 \leq \ldots \leq I_\kappa$*, such that* $M = I_\kappa$*, and*

1. $I_0 = \emptyset$.
2. *For successor ordinals* α *with* $0 < \alpha \leq \kappa$, I_α *is a minimal extension of* $I_{\alpha-1}$ *satisfying all rules which are applicable in* $[I_{\alpha-1}, M]$.
3. *For limit ordinals* $\lambda \leq \kappa$, $I_\lambda = \sup_{\alpha<\lambda} I_\alpha$.

We say that M *is* generated *by the* R-stable chain $I_0 \leq \ldots \leq I_\kappa$.

We also say simply 'stable' instead of 'stable generated' model. Stable entailment is defined as follows:

$$R \models_s F \quad \text{iff} \quad \mathrm{Mod}_s(R) \subseteq \mathrm{Mod}(\{F\})$$

where $F \in L^0(\sigma)$.

Example 2 (continued) *Only* M_1 *is a stable model of* R_2 *generated by the stable chain* $\emptyset \leq \{p(c){:}vl\} \leq \{p(c){:}vl, r(c){:}vl\}$. M_2 *is not a minimal extension of* $I_1 = \{p(c){:}vl\}$ *satisfying* Hr *for all* $r \in (R_2)_{[I_1,M_2]}$*, simply because* $(R_2)_{[I_1,M_2]} = \emptyset$*, i.e. the extension condition is trivially satisfied, and hence* I_1 *is the only minimal extension of* I_1*. Consequently,* $\mathrm{Mod}_s(R_2) = \{M_1\}$*, and hence* $R_2 \models_s r(c){:}vl$.

Stable models do not exist in all cases. For instance, $R = \{p \leftarrow \neg p\}$ has exactly one minimal model, $\{p{:}ll\}$, which is not stable, however. We call a program without stable models *unstable*.

Observation 7 *Minimal and stable models coincide for non-disjunctive fuzzy logic programs without negation.*

This observation applies in particular to those negation-free fuzzy logic programs proposed in [3, 1, 4, 9].

5.2 Embedding Normal Logic Programs

Recall that a *normal logic program* consists of rules of the form

$$a \leftarrow l_1 \wedge \ldots \wedge l_n$$

where a stands for an atom, and $l_i = a_i | \neg a_i$ for a literal. In an alternative notation, commas are used instead of the conjunction symbol $\wedge$.

For $B \subseteq \mathrm{Lit}$, let B^- denote the set of atoms which occur negated in B, i.e. $B^- = \{a \in \mathrm{At} \mid \neg a \in B\}$, and let $B^+ = \{a \in \mathrm{At} \mid a \in B\}$. It holds that for any $B \subseteq \mathrm{Lit}^0$, and any Herbrand interpretation $I \subseteq \mathrm{At}^0$,

$$I \models B \quad \text{iff} \quad B^+ \subseteq I \ \&\ B^- \cap I = \emptyset$$

Definition 13 (Gelfond/Lifschitz 1988) *Let* Π *be a normal logic program, and* $I \subseteq \mathrm{At}$*. Then the* Gelfond-Lifschitz transformation *of* Π *with respect to* I *is defined as*

$$\Pi^I = \{a \leftarrow B^+ \mid (a \leftarrow B) \in \Pi, \text{ and } B^- \cap I = \emptyset\}$$

and the Gelfond-Lifschitz operator Γ_Π *is defined as follows:* $\Gamma_\Pi(I)$ *denotes the unique minimal model of* Π^I*, i.e.* $\Gamma_\Pi(I) = M_{\Pi^I}$*. Fixpoints of* Γ_Π *are called* stable models *of* Π*. According to Gelfond and Lifschitz, an atom* a *follows from* Π*, symbolically* $\Pi \vdash a$*, if it is satisfied in all stable models of* Π.

A normal logic program Π can be faithfully embedded in a fuzzy logic program Π^f by setting

$$\begin{array}{rcl} \Pi^f & = & \{a{:}1 \leftarrow l_1{:}1 \wedge \ldots \wedge l_n{:}1 \mid \\ & & (a \leftarrow l_1 \wedge \ldots \wedge l_n) \in \Pi\} \end{array}$$

Claim 2 *Fuzzy logic programs are a conservative extension of normal logic programs, i.e.*

$$\Pi^f \models_s a{:}1 \ \textit{whenever}\ \Pi \vdash a$$

6 Related Work

In [7], the distinction between uncertainty 'at the tuple level' and uncertainty 'at the data-value level' in databases is made. However, no fuzzy inference relation including a logical treatment of negative evidence and negation in database queries is defined. An axiomatic definiton of relational databases with fuzzy-set-valued attributes along the lines of Reiter's database completion theory is proposed in [8].

The main problem for many fuzzy logic programming approaches, such as [3, 4, 9], is the fuzzy logic evaluation of negation: $C(\sim F) = 1 - C(F)$, which violates the law of the excluded contradiction and is not compatible with the intended semantics of negation in logic programs. As a consequence of this problem, neither of these approaches allows for negation in the body of a rule. We have remedied the negation problem of fuzzy logic by proposing our *semi-possibilistic* logic where we combine the min/max-evaluation of conjunction and disjunction with a supraintuitionistic evaluation of negation, thereby preserving the law of the excluded contradiction and providing a semantic link to negation-as-failure. By defining a satisfaction relation for uncertainty-qualified formulas, we show that the evaluation of conjunction and disjunction is not a matter of choice, as suggested by the t-norm approach to fuzzy logic (adopted, e.g., in [4, 9]), but has to be done by means of minimum and maximum.

7 Conclusion

We have shown how to combine uncertainty in the form of fuzzy relations with the semantic framework of logic programming, i.e. with minimal and stable Herbrand models. Our model of uncertain inference is derived from the possibility theory of Zadeh, Dubois and Prade. But unlike the rather unsatisfactory treatment of negation in fuzzy logic, and the non-compositional semantics of Dubois and Prade's possibilistic logic, we propose a more logical account of handling negative evidence and negation in our compositional semi-possibilistic logic.

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