# All finitely axiomatizable subframe logics containing the provability logic $\mathbf{CSM}_0$ are decidable

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**Abstract**

In this paper we investigate those extensions of the bimodal provability logic $\mathbf{CSM}_0$ (alias $\mathbf{PRL}_1$ or $\mathbf{F}^-$) which are subframe logics, i.e. whose general frames are closed under a certain type of substructures. Most bimodal provability logics are in this class. The main result states that all finitely axiomatizable subframe logics containing $\mathbf{CSM}_0$ are decidable. We note that, as a rule, interesting systems in this class do not have the finite model property and are not even complete with respect to Kripke semantics.

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## 1 Introduction

The aim of this paper is to give a proof of the decidability of all finitely axiomatizable quasi-normal subframe logics containing the bimodal provability logic $\mathbf{CSM}_0$. A subframe logic is a logic whose frames are closed under a certain type of substructures. All the provability logics investigated by A. Visser [8] and hence a number of systems from [7], [4], [1] and [2] are included in this class. We shall, however, not discuss the interpretation by provability predicates in this paper, but refer the reader to those papers for more information.

The decidability of all bimodal provability logics investigated in the literature is known. So the interest of the present investigation does not lie in the proof of the decidability of a specific sytem but in the fact that it delivers a uniform proof of the decidability of a large class of logics which is defined by means of a closure condition (namely, closure under taking certain substructures) on the frame classes definable by them.

Normal unimodal subframe logics containing **K4** have been introduced and investigated by K. Fine [3]. Among them are the provability logic **GL** and the logics **S4** and **S4.3**. The main result of [3] states that all normal subframe logics containing **K4** have the finite model property. So finitely axiomatizable ones are decidable. A theory of both uni- and polymodal subframe logics is developed in [9]. It turned out that subframe logics in general are quite complex. For example, there exist undecidable finitely axiomatizable uni-modal subframe logics. So the positive result of the present paper is rather surprising. On our way to this theorem we shall establish some results which are of independent interest: Firstly we deliver axiomatizations by means of canonical formulas which are similar to those introduced in [12] for logics containing **K4** and those introduced in [11] for tense logics. Then we prove completeness of all quasi-normal subframe logics containing $\mathbf{CSM}_0$ with respect to rather simple (descriptive) frames.

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## 2 Preliminaries

Denote by $\mathcal{L}_2$ the propositional language with classical connectives $\wedge$ and $\neg$ and modal operators $\Box_1$ and $\Box_2$. In this paper we call a subset $\Lambda$ of $\mathcal{L}_2$ are *normal csm-logic* iff it contains

A1 All classical tautologies.

A2 $\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$, for $i \in \{1, 2\}$.

A3 $\Box_i p \to \Box_i \Box_i p$, for $i \in \{1, 2\}$.

A4 $\Box_i(\Box_i p \to p) \to \Box_i p$, for $i \in \{1, 2\}$.

A5 $\Box_1 p \to \Box_2 p$.

A6 $\Box_2 p \to \Box_1 \Box_2 p$.

and is closed under modus ponens: $p, p \to q/q$, substitutions and $p/\Box_i p$, $i \in \{1, 2\}$. The smallest normal csm-logic is denoted by $\mathbf{CSM}_0$ and the smallest normal csm-logic containing a normal csm-logic $\Lambda$ and a set of formulas $\Gamma$ is denoted by $\Lambda \oplus \Gamma$. We get, for example, the following logics from e.g. [8].

- $\mathbf{CSM}_1 = \mathbf{CSM}_0 \oplus \Box_2(\Box_1 p \to p)$. (This is $\mathbf{PRL}_{ZF}$ in [7] and **F** in [4].)
- $\mathbf{NB}_1 = \mathbf{CSM}_0 \oplus (\neg\Box_1 p \wedge \Box_2 p) \to \Box_2(\Box_1 q \to q)$.

A *quasi-normal csm-logic* is a subset of $\mathcal{L}_2$ which contains $\mathbf{CSM}_0$ and is closed under modus ponens and substitutions. The smallest quasi-normal csm-logic containing a quasi-normal csm-logic $\Lambda$ and a set of formulas $\Gamma$ is denoted by $\Lambda + \Gamma$. Examples of logics which are quasi-normal but not normal are the following systems from e.g. [8].

- $\mathbf{CSM}_2 = \mathbf{CSM}_1 + \Box_1 p \to p$. (This is $\mathbf{PRL}_{ZF}$ + Reflection$_{\Box_1}$ in [7] and $\mathbf{F}_1$ in [4].)
- $\mathbf{CSM}_3 = \mathbf{CSM}_2 + \Box_2 p \to p$. (This is $\mathbf{PRL}_{ZF}$ + Reflection$_{\Box_2}$ in [7].)
- $\mathbf{NB}_2 = \mathbf{NB}_1 + \Box_2 p \to p + \Box_2 p \to \Box_1 p$.

We are now going to introduce the structures in which the modal language is interpreted. A structure $\mathcal{G} = \langle W, R_1, R_2, P\rangle$ is a *frame* if $R_1$ and $R_2$ are transitive relations on $W$ satisfying $R_2 \subseteq R_1$ and $\mathbf{Con} = R_1 \circ R_2 \subseteq R_2$, i.e.,

$$\forall x \forall y \forall z (x R_1 y \wedge y R_2 z \to x R_2 z),$$

and $P$ is a set of subsets of $W$ containing $W$ and closed under intersection, complement and

$$\Box_i a = \{y \in W : \forall z \in W (y R_i z \to z \in P\},$$

for $i \in \{1, 2\}$. Notice that $R_1 \supseteq R_2$ corresponds to [A5] and $\mathbf{Con}$ corresponds to [A6]. $\mathcal{G}$ is *refined* if moreover

$$\forall x \in W, y \in W (x = y \leftrightarrow (\forall a \in P)(x \in a \leftrightarrow y \in a)).$$

$$\forall x \in W, y \in W (x R_i y \leftrightarrow (\forall a \in P)(x \in \Box_i a \to y \in a)),$$

for $i \in \{1, 2\}$. $\mathcal{G}$ is *descriptive* iff it is refined and $\bigcap U \neq \emptyset$, for each ultrafilter $U$ in $P$. A *Kripke frame* is a frame $\langle W, R_1, R_2, P\rangle$ in which $P$ is the powerset of $W$. In this case we shall mostly omit writing $P$. So, the *underlying Kripke frame* of a frame $\mathcal{G} = \langle W, R_1, R_2, P\rangle$ is $\langle W, R_1, R_2\rangle$. A *valuation* $\beta$ in $\mathcal{G}$ is a homomorphism from the algebra of formulas into $P$ (equiped with the usual operations). We write

- $\langle \mathcal{G}, \beta, x\rangle \models \varphi$ iff $x \in \beta(\varphi)$.
- $\langle \mathcal{G}, x\rangle \models \varphi$ iff $\langle \mathcal{G}, \beta, x\rangle \models \varphi$, for all $\beta$.
- $\langle \mathcal{G}, \beta\rangle \models \varphi$ iff $\beta(\varphi) = W$.
- $\mathcal{G} \models \varphi$ iff $\langle \mathcal{G}, \beta\rangle \models \varphi$, for all $\beta$.

The logic *determined* by $\mathcal{G}$ is the set of all $\varphi$ such that $\mathcal{G} \models \varphi$. A pair $\langle \mathcal{G}, x\rangle$ with $x \in W$ and $\mathcal{G}$ a frame is called a *pointed* frame. The logic determined by a pointed frame $\langle \mathcal{G}, x\rangle$ is the set of all $\varphi$ with $\langle \mathcal{G}, x\rangle \models \varphi$. Recall the following well known completeness result for modal logics (cf. e.g. [5] and [6]).

**Theorem 1** *Each normal csm-logic is determined by a class of descriptive frames. Each quasi-normal csm-logic is determined by a class of descriptive pointed frames $\langle \mathcal{G}, x\rangle$ such that $\mathcal{G} \models \mathbf{CSM}_0$.*

Consider a frame $\mathcal{G} = \langle W, R_1, R_2, P\rangle$ and a set $a \in P$. Then

$$\mathcal{G}_a = \langle a, R_1 \cap a \times a, R_2 \cap a \times a, \{a \cap b : b \in P\}\rangle$$

is a frame as well and we call it a *subframe* of $\mathcal{G}$. A normal csm-logic $\Lambda$ is called a *subframe logic* iff the class of frames validating $\Lambda$ is closed under forming subframes. For a pointed frame $\langle \mathcal{G}, x\rangle$ we call $\langle \mathcal{G}_a, x\rangle$, $a \in P$, a *pointed subframe* of $\mathcal{G}$ whenever $x \in a$. A quasi-normal csm-logic is called a subframe logic iff the class of pointed frames validating $\Lambda$ is closed under forming pointed subframes. It is easily checked that both definitions coincide for quasi-normal logics which are already normal. The following Theorem is proved in [9].

**Theorem 2** *(i) A normal csm-logic is a subframe logic iff it is determined by a class of frames closed under forming subframes. The normal subframe logics containing* $\mathbf{CSM}_0$ *form a complete sublattice of the lattice of normal csm-logic.*

*(ii) A quasi-normal csm-logic is a subframe logic iff it is determined by a class of pointed frames closed under forming pointed subframes. The quasi-normal subframe logics containing* $\mathbf{CSM}_0$ *form a complete sublattice of the lattice of quasi-normal csm-logics.*

We refer the reader to [3] and [13] for more information on (unimodal) subframe logics containing **K4** and to [9] for information on subframe logics in general. Notice that the unimodal logic **GL** (which may be defined as the unimodal fragment of $\mathbf{CSM}_0$) is a subframe logic (cf. [3]). Hence $\mathbf{CSM}_0$ is easily seen to be a subframe logic.

# 3 The Results

Some notation is required in order to formulate the results. For $i \in \{1, 2\}$ we write $xR_i^w y$ iff $xR_i y$ or $x = y$. On the other hand, we write $xR_i^p y$ iff $xR_i y$ and $\neg(yR_i x)$. A *$R_i$-cluster* is a non-empty set $C$ of the form $C = \{x : xR_i y \wedge yR_i x\}$. (Notice that this notation is not standard. Here irreflexive points are not called cluster. So what is here just called cluster is usually called non degenerate clusters.) For an $R_i$-cluster $C$ we use the notation $CR_i y$, $yR_i C$, $CR_i D$ in the obvious way. We also write $CR_2 x$ whenever $C$ is an $R_1$-cluster and there exists $y \in C$ with $yR_2 x$. This is justified by condition **Con**, since we can infer $zR_2 x$, for all $z \in C$. We shall use this fact rather often.

Call a frame $\mathcal{G}$ *rooted* if there exists an $r \in W$ such that $W = \{y \in W : rR_1^w y\}$. Then $r$ is called a *root* of $\mathcal{G}$. A frame $\langle W, R_1, R_2, \rangle$ is called a *surrogate frame* iff it is finite and has precisely one root $r$ and all points different from $r$ are $R_2$-irreflexive. A *normal surrogate frame* $\langle W, R_1, R_2\rangle$ is a surrogate frame in which the root $r$ is $R_1$-irreflexive. With each surrogate frame $\mathcal{G} = \langle W, R_1, R_2 \rangle$ we associate the formula

$$\begin{aligned}
\delta(\mathcal{G}) \quad = \quad & \bigwedge \langle (p_x \to \Diamond_1 p_y) | x R_1^p y; x, y \in W \rangle \wedge \\
& \bigwedge \langle p_x \to \Diamond_2 p_y | x R_2^p y; x, y \in W \rangle \wedge \\
& \bigwedge \langle p_x \to \neg p_y | x \neq y; x, y \in W \rangle \wedge \\
& \bigwedge \langle p_x \to \neg \Diamond_1 p_y | \neg (x R_1 y); x, y \in W \rangle \wedge \\
& \bigwedge \langle p_x \to \neg \Diamond_2 p_y | \neg (x R_2 y); x, y \in W \rangle
\end{aligned}$$

and put for the root $r$ of $\mathcal{G}$,

$$\alpha(\mathcal{G}) = \delta(\mathcal{G}) \wedge \Box_1 \delta(\mathcal{G}) \to \neg p_r.$$

Notice that $\alpha(\mathcal{G})$ is the splitting formula (or subframe formula) associated with $\mathcal{G}$ in the lattice of normal subframe logics containing $\mathbf{CSM}_0$ whenever $\mathcal{G}$ contains no $R_1$-cluster. Here we call $\varphi$ the splitting formula associated with $\mathcal{G}$ if $\mathbf{CSM}_0 \oplus \varphi$ is the smallest normal subframe logic containing $\mathbf{CSM}_0$ which is refuted in $\mathcal{G}$.

We now briefly explain the meaning of the formulas $\alpha(\mathcal{G})$ in general. Given a frame $\mathcal{H} = \langle V, S_1, S_2, Q \rangle$ validating $\mathbf{CSM}_0$ we say that a mapping $h$ from $V$ onto $W$ is a *weak reduction* of $\mathcal{H}$ to $\mathcal{G}$ if for $i \in \{1, 2\}$ and all $x, y \in V$,

- $x S_i y$ implies $h(x) R_i h(y)$,
- $h(x) R_i^p h(y)$ implies $\exists z \in V (x S_i z \wedge h(z) = h(y))$,
- $h^{-1}(a) \in Q$, for all $a \subseteq W$.

The standard definition of a reduction (alias p-morphism) is relaxed here in the second condition. A frame $\mathcal{H}$ is said to be *weakly subreducible* to a surrogate frame $\mathcal{G}$ if a subframe of $\mathcal{H}$ is weakly reducible to $\mathcal{G}$. The following lemma explains the meaning of the canonical formulas by means of weak subreductions.

**Lemma 3** *For each surrogate frame $\mathcal{G}$ and each $\mathbf{CSM}_0$-frame $\mathcal{H}$, $\mathcal{H} \not\models \alpha(\mathcal{G})$ iff $\mathcal{H}$ is weakly subreducible to $\mathcal{G}$.*

We leave the straightforward proof of this Lemma to the reader, since it will not be required in what follows. However, using it one can is easily check that $\mathbf{CSM}_0 \oplus \alpha(\mathcal{G})$ and $\mathbf{SCM}_0 + \alpha(\mathcal{G})$ are always subframe logics. Conversely, we have

**Theorem 4** *(i) There is an algorithm which, given a formula $\varphi$ such that $\mathbf{CSM}_0 + \varphi$ is a subframe logic, returns surrogate frames $\mathcal{G}_1, \ldots, \mathcal{G}_n$ such that*

$$\mathbf{CSM}_0 + \varphi = \mathbf{CSM}_0 + \alpha(\mathcal{G}_1) + \ldots + \alpha(\mathcal{G}_n).$$

*There is an algorithm which, given a formula $\varphi$ such that $\mathbf{CSM}_0 \oplus \varphi$ is a subframe logic, returns normal surrogate frames $\mathcal{G}_1, \ldots, \mathcal{G}_n$ such that*

$$\mathbf{CSM}_0 \oplus \varphi = \mathbf{CSM}_0 \oplus \alpha(\mathcal{G}_1) \oplus \ldots \oplus \alpha(\mathcal{G}_n).$$

The proof will be delivered in section 4.

**Example 5** All the logics introduced above are subframe logics. We have the following axiomatizations.

- $\mathbf{CSM}_1 = \mathbf{CSM}_0 \oplus \alpha(\langle\{0,1\},\{(0,1)\},\{(0,1)\}\rangle)$.
- $\mathbf{CSM}_0 + \Box_1 p \to p = \mathbf{CSM}_0 + \alpha(\langle\{0\},\emptyset,\emptyset\rangle)$.
- $\mathbf{CSM}_0 + \Box_2 p \to p = \mathbf{CSM}_0 + \alpha(\langle\{0\},\emptyset,\emptyset\rangle) + \alpha(\langle\{0\},\{(0,0)\},\emptyset\rangle)$.
- $\mathbf{CSM}_0 + \Box_2 p \to \Box_1 p = \mathbf{CSM}_0 + \alpha(\langle\{0,1\},\{(0,0),(0,1),(1,1)\},\{(0,0)\}\rangle)$.
- $$\begin{aligned}\mathbf{NB}_1 \;=\;& \mathbf{CSM}_0 \oplus \alpha(\langle\{0,1,2\},\{(0,1),(0,2),(1,2)\},\{(0,1)\}\rangle) \oplus \\ & \alpha(\langle\{0,1,2\},\{(0,1),(0,2),(1,2)\},\{(0,2),(1,2)\}\rangle) \oplus \\ & \alpha(\langle\{0,1,2\},\{(0,1),(0,2),(1,2)\},\{(0,2)\}\rangle) \oplus \\ & \alpha(\langle\{0,1,2\},\{(0,1),(0,2)\},\{(0,1)\}\rangle).\end{aligned}$$

We prove the correctness of the equation $\mathbf{CSM}_0 + \Box_2 p \to \Box_1 p = \mathbf{CSM}_0 + \alpha$, where $\alpha = \alpha(\langle\{0,1\},\{(0,0),(0,1),(1,1)\},\{(0,0)\}\rangle)$. Clearly $\mathbf{CSM}_0 + \Box_2 p \to \Box_1 p$ contains $\alpha$. So it suffices to show for all descriptive $\langle\mathcal{H},r\rangle = \langle\langle V,S_1,S_2,Q\rangle,r\rangle$, $\langle\mathcal{H},r\rangle \not\models \Box_2 p \to \Box_1 p$ implies $\langle\mathcal{H},r\rangle \not\models \alpha$. The condition $\langle\mathcal{H},r\rangle \not\models \Box_2 p \to \Box_1 p$ implies that there exists $x \in V$ such that $rR_1x$ but $\neg(rR_2x)$.

Case 1. $x \neq r$. By **Con** we have $\neg(xR_2x)$ and so it is easily checked that $\langle\mathcal{H},r\rangle \not\models \alpha$.

Case 2. $x = r$. We reduce this case to the first one. By $\neg(rR_2r)$ we find $a \in Q$ such that $r \in b = a \cap \neg\Diamond_2 a$. On the other hand there exists $y \in b$ different from $r$ such that $rR_1y$, since $r$ is $R_1$-reflexive. We do not have $rR_2y$, since $y \in b$.

The other axiomatizations are proved similarly and are left to the reader.

A quasi-normal csm-logic $\Lambda$ is called *finitely axiomatizable* iff there exists a finite set of formulas $\Gamma$ such that $\Lambda = \mathbf{CSM}_0 + \Gamma$. Notice that a normal csm-logic is finitely axiomatizable iff there exists a finite set of formulas $\Gamma$ such that $\Lambda = \mathbf{CSM}_0 \oplus \Gamma$. This follows immediately from the easily proved fact that

$$\mathbf{CSM}_0 \oplus \Gamma = \mathbf{CSM}_0 + \Gamma + \{\Box_1\varphi : \varphi \in \Gamma\}.$$

**Theorem 6** *(i) All finitely axiomatizable normal subframe logics containing* $\mathbf{CSM}_0$ *are decidable.*

*(i) All finitely axiomatizable quasi-normal subframe logics containing* $\mathbf{CSM}_0$ *are decidable.*

The proof will be delivered in the last section. It is based on a rather strong completeness result for subframe logics. To formulate this result we have to manipulate some frames. In what follows we shall assume that for each surrogate frame $\mathcal{G}$ and each $R_1$-cluster $C$ of cardinality $m = |C|$ we have a fixed enumeration

$$C = \{j(C) : j < m\}$$

of the elements of $C$.

Suppose that $\mathcal{G} = \langle W, R_1, R_2\rangle$ is a surrogate frame and let $\alpha \in \omega + 1$ (i.e., $\alpha \in \omega$ or $\alpha = \omega$). With $\mathcal{G}$ we shall associate a finite set of frames $\mathrm{Ext}_\alpha\mathcal{G} = \{\mathcal{G}^\alpha_{\vec{A}} : \vec{A} \in \mathrm{Seq}\mathcal{G}\}$ ($\mathrm{Seq}\mathcal{G}$ will be defined below). Let us first assume that the root $r$ of $\mathcal{G}$ is $R_2$-irreflexive. Then, roughly speaking, the frames in $\mathrm{Ext}_\alpha\mathcal{G}$ are the results when we insert an $R_1$-chain $C[\alpha]$ of length $\alpha$ between each $R_1$-cluster $C$ and its successors. As concerns the relation $R_1$ there will be only one way to do this. We get (a finite) set of frames $\mathrm{Ext}_\alpha\mathcal{G}$ since a point which $R_2$-sees a point in $C$ need not (but may) $R_2$-see certain points in the chain $C[\alpha]$. Here is the formal definition: Denote, for each $R_1$-cluster $C$ by $C[\alpha]$ the set $\{(n, C) : n \in \alpha\}$. Mostly we shall write $n_C$ for $(n, C)$. Denoting elements of $C$ by $n(C)$, $n < |C|$, and elements of $C[\alpha]$ by $n_C$ (or $(n, C)$) will turn out to be quite convenient. We hope that the similarity does not lead to confusion. Define the set $\mathrm{Seq}\mathcal{G}$ as follows:

$\mathrm{Seq}\mathcal{G}$ consists of all sequences $\vec{A}$ of the form

$$\vec{A} = \langle A_x : xR_1x\rangle,$$

where $A_x \subseteq \{y \in W - C : yR_2x\}$ satisfies the following closure condition for all $y, z \in W$:

$$y \in A_x \text{ and } zR_1y \text{ imply } z \in A_x.$$

For each $\vec{A} \in \mathrm{Seq}\mathcal{G}$ and $\alpha \in \omega + 1$ the structure $\mathcal{G}^\alpha_{\vec{A}} = \langle V, S_0, S_1\rangle$ is the (uniquely determined) frame satisfying the following conditions:

- $V = W \cup \bigcup\{C[\alpha] : C \text{ a } R_1\text{-cluster in } \mathcal{G}\}$.
- $R_i = S_i \cap (W \times W)$, for $i \in \{1, 2\}$.
- The $R_i$-clusters coincide with the $S_i$-clusters, for $i \in \{1, 2\}$.

and the following conditions for $S_1$: For all $R_1$-clusters $C$,

1. for all $x \in C[\alpha]$ : $CS_1x$,

2. for all $n_C, m_C \in C[\alpha]: \ n_C S_1 m_C$ iff $n > m$,
3. for all $y \in W - C$ and $x \in C[\alpha]$: $xS_1y \Leftrightarrow CR_1y$ and $yS_1x \Leftrightarrow yR_1C$,
4. for all $x \in C[\alpha]$ and $y \in V - W : \ xS_1y \Leftrightarrow CS_1y$.

and the following conditions for $S_2$: For all $R_1$-clusters $C$,

1. $((C[\alpha] \cup C) \times (C[\alpha] \cup C)) \cap S_2 = \emptyset$,
2. for all $y \in W - C$ and $x \in C[\alpha]$: $xS_2y \Leftrightarrow CR_2y$,
3. for all $y \in W - C$ and $C = \{j(C) : j < m\}$ and $x \in C[\alpha]$:
$$(yS_2x) \quad \Leftrightarrow \quad (\exists i \in \omega)(\exists j < m)(x = (im + j, C) \wedge y \in A_{j(C)}),$$
4. for all $x \in C[\alpha]$ and $y \in V - W : \ xS_2y \Leftrightarrow CS_2y$.

**Example 7** Let $\mathcal{G} = \langle \{0,1\}, \{(0,1),(1,1)\}, \{(0,1)\} \rangle$. Then $\{1\}$ is the only $R_1$-cluster in $\mathcal{G}$ and Seq$\mathcal{G} = \{\langle \emptyset \rangle, \langle \{0\} \rangle\}$. Hence $\text{Ext}_\omega(\mathcal{G}) = \{\mathcal{G}^\omega_{\langle \emptyset \rangle}, \mathcal{G}^\omega_{\langle \{0\} \rangle}\}$ for
$$\mathcal{G}^\omega_{\langle \emptyset \rangle} = \langle V, S_1, S_2 \rangle \text{ and } \mathcal{G}^\omega_{\langle \{0\} \rangle} = \langle V, S_1, S_2' \rangle,$$
where
$$V = \{0,1\} \cup \{(n,1) : n \in \omega\},$$
$$S_1 = \{(0,1),(1,1)\} \cup \{(m,(n,1)) : n \in \omega, m = 0,1\} \cup \{((n,1),(m,1)) : n > m\},$$
$$S_2 = \{(0,1)\} \text{ and } S_2' = S_2 \cup \{(0,(n,1)) : n \in \omega\}.$$

Suppose now that the root $r$ of $\mathcal{G} = \langle W, R_1, R_2 \rangle$ is $R_2$-reflexive. Put $D = \{r\}$. We define $\mathcal{G}^\alpha_{\vec{A}}$ by inserting an $R_2$-chain $D[\alpha]$ between $r$ and its successors. More precisely, define for $\vec{A} \in \text{Seq}\mathcal{G}$ the frame $\mathcal{G}^\alpha_{\vec{A}} = \langle V', S_1', S_2' \rangle$ as follows: First form $\mathcal{F}^\alpha_{\vec{A}} = \langle V, S_1, S_2 \rangle$ for the surrogate frame $\mathcal{F} = \langle W, R_1 - \{(r,r)\}, R_2 - \{(r,r)\} \rangle$ as above and then put
$$V' = V \cup D[\alpha], \text{ where } D[\alpha] = \{n_D : n \in \alpha\}$$
$$\begin{aligned} S_i' \quad = \quad & S_i \cup \{(r,r)\} \cup \{(n_D, m_D) : n > m\} \cup \\ & \{(x,y) : x \in D[\alpha], y \in V, xS_iy\} \cup \\ & D \times D[\alpha], \end{aligned}$$
for $i = 1, 2$.

**Example 8** Let $\mathcal{G}_1 = \langle \{0\}, \{(0,0)\}, \emptyset \rangle$ and $\mathcal{G}_2 = \langle \{0\}, \{(0,0)\}, \{(0,0)\} \rangle$. In both cases, Seq$\mathcal{G}$ consists only of the empty set. We get
$$(\mathcal{G}_1)^\omega_\emptyset = \langle \{0\} \cup \{(n,0) : n \in \omega\}, S, \emptyset \rangle \text{ and } (\mathcal{G}_2)^\omega_\emptyset = \langle \{0\} \cup \{(n,0) : n \in \omega\}, S, S \rangle,$$
where $S = \{(0,0)\} \cup \{(0,(n,0)) : n \in \omega\} \cup \{((n,0),(m,0)) : n > m\}$.

Clearly, for $\alpha \neq \omega$ all the frames in $\mathrm{Ext}_\alpha \mathcal{G}$ are finite, but the frames $\mathcal{G}^\omega_{\vec{A}}$ are infinite. In order to formulate the completeness result we define descriptive frames based on $\mathcal{G}^\omega_{\vec{A}}$. Namely, define for $\mathcal{G}^\omega_{\vec{A}} = \langle V, S_1, S_2 \rangle$ the frame $\mathbf{G}[\mathcal{G}^\omega_{\vec{A}}] = \langle V, S_1, S_2, P \rangle$ as follows: For each $R_1$-cluster $C = \{j(C) : j < m\}$ in $\mathcal{G}$ let

$$P_C = \{\{j(C)\} \cup \{(im + j, C) : i \in \omega\} : j = 0, \ldots, m-1\}.$$

Then $P$ denotes the closure of

$$\{\{x\} : \neg(xS_1x)\} \cup \{P_C : \ C \text{ is a } R_1\text{-cluster in } \mathcal{G}\}$$

under intersections and complements. The proof of the following result is straightforward but a bit tedious.

**Theorem 9** *For all surrogate frames $\mathcal{G}$, all structures in $\{\mathbf{G}[\mathcal{G}^\omega_{\vec{A}}] : \vec{A} \in Seq\mathcal{G}\}$ are descriptive frames.*

The following completeness result will be proved in section 5.

**Theorem 10** *(i) Each normal subframe logic $\Lambda$ containing $\mathbf{CSM}_0$ is determined by a set of frames of the form $\mathbf{G}[\mathcal{G}^\omega_{\vec{A}}]$, where $\mathcal{G}$ is a normal surrogate frame and $\vec{A} \in Seq\mathcal{G}$. Moreover, for each $\varphi \notin \Lambda$ there exists a normal surrogate frame $\mathcal{G}$ of cardinality $\leq \sum_{i=0}^{2k-1}(2k)^i$ such that $\mathbf{G}[\mathcal{G}^\omega_{\vec{A}}] \models \Lambda$ and $\mathbf{G}[\mathcal{G}^\omega_{\vec{A}}] \not\models \varphi$, for an $\vec{A} \in Seq\mathcal{G}$. Here $k = |\mathbf{Sub}\varphi|$.*

*(ii) Each quasi-normal subframe logic containing $\mathbf{CSM}_0$ is determined by a set of pointed frames of the form $\langle \mathbf{G}[\mathcal{G}^\omega_{\vec{A}}], r \rangle$, where $\mathcal{G}$ is a surrogate frame with root $r$ and $\vec{A} \in Seq\mathcal{G}$. Moreover, for each $\varphi \notin \Lambda$ there exists a surrogate frame $\mathcal{G}$ of cardinality $\leq \sum_{i=0}^{2k}(2k)^i$ such that $\langle \mathbf{G}[\mathcal{G}^\omega_{\vec{A}}], r \rangle \models \Lambda$ and $\langle \mathbf{G}[\mathcal{G}^\omega_{\vec{A}}], r \rangle \not\models \varphi$, for an $\vec{A} \in Seq\mathcal{G}$ and the root $r$ of $\mathcal{G}$. Here $k = |\mathbf{Sub}\varphi|$.*

# 4 Canonical Formulas

Let $\mathcal{G} = \langle W, R_1, R_2, P \rangle$ be a frame, $x \in W$ and $b \in P$. Then $x$ is called *$R_i$-maximal in* $b$, in symbols $x \in \max_{R_i}(b)$, if $x \in b$ but there does not exist a $y \in b$ with $xR_i^p y$. The following Lemma states the characteristic property of frames validating [A4]. For a proof consult e.g. [3].

**Lemma 11** *Let $\mathcal{G} = \langle W, R_1, R_2, P \rangle$ be descriptive and assume $\mathcal{G} \models \mathbf{CSM}_0$. For all $b \in P$ and $i \in \{1, 2\}$ the following holds: (i) All $x \in \max_{R_i}(b)$ are $R_i$-irreflexive. (ii) For all $y \in b$ there exists a $x \in \max_{R_i}(b)$ with $yR_i^w x$.*

Denote by $\mathbf{Sub}\varphi$ the set of subformulas of a formula $\varphi$. A valuation $\beta$ of a surrogate frame $\mathcal{G} = \langle W, R_1, R_2 \rangle$ is *$\varphi$-good* if for each $\psi \in \mathbf{Sub}\varphi$ and each $R_i$-cluster $C$ with $\beta(\psi) \cap C \neq \emptyset$ there exists a $y \in \beta(\psi)$ with $CR_i^p y$.

**Lemma 12** *Let $\mathcal{H} = \langle W, R_1, R_2, P\rangle$ be descriptive and assume $\mathcal{H} \models$ **CSM**$_0$. Suppose $\langle \mathcal{H}, \beta, x\rangle \not\models \varphi$ and denote by $k$ the cardinality of* **Sub**$\varphi$.

*(i) There exists a normal surrogate frame $\mathcal{G}$ of cardinality $\leq \sum_{i=0}^{2k-1}(2k)^i$ which is a subframe of the underlying Kripke-frame of $\mathcal{H}$ such that there is a $\varphi$-good valuation $\beta^*$ of $\mathcal{G}$ which refutes $\varphi$.*

*(ii) There exists a surrogate frame $\mathcal{G}$ of cardinality $\leq \sum_{i=0}^{2k}(2k)^i$ which is a subframe of the underlying Kripke-frame of $\mathcal{H}$ containg $x$ such that there is a $\varphi$-good valuation $\beta^*$ of $\mathcal{G}$ with $\langle \mathcal{G}, \beta^*, x\rangle \not\models \varphi$.*

**Proof.** (i) By Lemma 11 we can take a $z \in \max_{R_1}(\beta(\neg\varphi))$. Define $V$ inductively as follows: First put $F_0 = \{z\}$. Suppose now that $F_n$ is defined. For each $y \in F_n$, $R \in \{R_1, R_2\}$, $\psi \in$ **Sub**$\varphi$ such that there exists $y' \in \beta(\psi)$ with $yRy'$ select a $y'' \in \max_R(\beta(\psi))$ with $yRy''$. Again, this is possible by Lemma 11. We take $y''$ from $F_n$ whenever possible. Denote by $D_{n+1}$ the set of new points selected in this way and put

$$F_{n+1} = F_n \cup D_{n+1} \text{ and } V = \bigcup\{F_n : n \in \omega\}.$$

We show that $F_{2k-1} = F_{2k}$. To this end suppose that $z' \in F_{2k} - F_{2k-1}$. There is a $R_1$-path $\langle x_j : 0 \leq j \leq 2k\rangle$ such that $z = x_0$ and $z' = x_{2k}$ and such that for each $x_{j+1}$ there exists a subformula $\psi$ of $\varphi$ and a $i \in \{1,2\}$ with $x_{j+1} \in \max_{R_i}(\beta(\psi))$ and $x_j R_i x_{j+1}$. But then there exists $\psi \in$ **Sub**$\varphi$ and $x_j, x_{j+l} \in \max_{R_2}(\beta(\psi))$ with $l > 0$ such that $x_{j+l-1} R_2 x_{j+l}$. Hence, by **Con**, $x_j R_2 x_{j+l}$. But this is impossible, since both $x_j$ as well as $x_{j+l}$ are in $\max_{R_2}(\beta(\psi))$.

Hence the cardinality of $V$ is bounded by $\sum_{i=0}^{2k-1}(2k)^i$. Define $\mathcal{G} = \langle V, S_1, S_2\rangle$, where $S_1$ and $S_2$ are the restrictions of $R_1$ and $R_2$ to $W$. $\mathcal{G}$ is a normal surrogate frame: Indeed, all points in $V$ are $R_2$-maximal, hence all points in $V$ are $S_2$-irreflexive, by Lemma 11. Also $z$ is $R_1$-maximal. Hence $z$ is $S_1$-irreflexive and the only root of $\mathcal{G}$.

Define a valuation $\beta^*$ of $\mathcal{G}$ by putting $\beta^*(p) = \beta(p) \cap V$, for all propositional variables $p$. One easily proves by induction

$$\beta^*(\psi) = V \cap \beta(\psi), \text{ for all } \psi \in \mathbf{Sub}\varphi.$$

Hence $\langle \mathcal{G}, \beta^*, x\rangle \not\models \varphi$ and $\beta^*$ is $\varphi$-good. So (i) is proved.

(ii) The construction of $\mathcal{G}$ is similar. This time, however, we put $F_0 = \{x\}$ and then proceed with the definition of $F_n$, $n > 0$, $\mathcal{G}$ and $\beta^*$ as above. Note that in this case $xR_2x$ or ($xR_1x$ and $\neg(xR_2x)$) is not excluded, and so $\mathcal{G}$ is possibly only a surrogate frame but not normal. ⊣

The $R_i$-*depth* of a point $x$ in a finite frame $\langle W, R_1, R_2\rangle$ is the length of the longest $R_i^p$-path $\langle x_i : 1 \leq i \leq n\rangle$ with $x = x_1$.

**Lemma 13** *Suppose that $\mathcal{H} = \langle W, R_1, R_2, P\rangle$ is descriptive and assume that $\mathcal{H} \models$ **CSM**$_0$. Suppose that $\mathcal{G} = \langle V, S_1, S_2\rangle$ is a surrogate frame which is subframe of the underlying Kripke frame of $\mathcal{H}$. Then there exists a valuation $\beta$ in $\mathcal{H}$ such that*

- *$x \in \beta(p_x)$, for all $x \in V$.*
- *$\langle \mathcal{H}, \beta \rangle \models \delta(\mathcal{G})$.*

**Proof.** For each pair $(x, y) \in V \times V$ with $x \neq y$ take $c_{x,y} \in P$ such that $x \in c_{x,y}$ and $y \notin c_{x,y}$. For each pair $(x, y) \in V \times V$ and $i \in \{1, 2\}$ with $\neg(xR_iy)$ take $b^i_{x,y} \in P$ such that $x \in \Box_i b^i_{x,y}$ and $y \notin b^i_{x,y}$. Sets with these properties exist since $\mathcal{H}$ is refined. Now we define $a_x$ by induction on the $R_1$-depth of $x$ in $\mathcal{G}$. Suppose that $a_y$ is defined for all $y$ of $R_1$-depth $\leq n$ and suppose that $x$ has $R_1$-depth $n + 1$. Put

$$\begin{aligned} a_x &= \bigcap\{c_{x,y} \cap -c_{y,x} : y \neq x; x, y \in V\} \cap \\ &\quad \bigcap\{\Box_i b^i_{x,y} : \neg(xR_iy); x, y \in V\} \cap \\ &\quad \bigcap\{-b^i_{y,x} : \neg(yR_ix); x, y \in V\} \cap \\ &\quad \bigcap\{\Diamond_i a_y : xR^p_i y; x, y \in V\} \end{aligned}$$

Define a valuation $\beta$ of $\mathcal{H}$ by putting $\beta(p_x) = a_x$, for all $x \in V$. It is easily checked that $\beta$ is as required. ⊣

**Lemma 14** *Suppose $\mathcal{H} = \langle W, R_1, R_2, P \rangle$ and $\mathcal{H} \models$ **CSM**$_0$. Let $\mathcal{G} = \langle V, S_1, S_2 \rangle$ be a surrogate frame with root $r$ and assume $\langle \mathcal{H}, z \rangle \not\models \alpha(\mathcal{G})$ and $\langle \mathcal{G}, \beta, r \rangle \not\models \varphi$, for a $\varphi$-good valuation $\beta$. Then there is a pointed subframe $\langle \mathcal{H}', z \rangle$ of $\langle \mathcal{H}, z \rangle$ such that $\langle \mathcal{H}', z \rangle \not\models \varphi$.*

**Proof.** Suppose that $\langle \mathcal{H}, \gamma, z \rangle \models \delta(\mathcal{G}) \wedge \Box_1 \delta(\mathcal{G}) \wedge p_r$. Define

$$b = \bigcup\{\gamma(p_x) : x \in V\}$$

and define a valuation $\gamma^*$ of $\mathcal{H}' = \mathcal{H}_b$ by putting

$$\gamma^*(p) = \bigcup\{\gamma(p_x) : x \in \beta(p)\}.$$

One easily proves by induction (by using that $\beta$ is $\varphi$-good)

$$\gamma^*(\psi) = \bigcup\{\gamma(p_x) : x \in \beta(\psi)\}, \text{ for all } \psi \in \mathbf{Sub}\varphi \ .$$

Hence $\langle \mathcal{H}', \gamma^*, z \rangle \not\models \varphi$. ⊣

**Proof of Theorem 4**

(ii) Suppose that $\mathbf{CSM}_0 \oplus \varphi$ is a subframe logic containing $\mathbf{CSM}_0$. Put $k = |\mathbf{Sub}\varphi|$. Let $\mathcal{G}_i$, $1 \leq i \leq n$, be the collection of normal surrogate frames satisfying the following conditions:

- The cardinality of $\mathcal{G}_i$ is $\leq \sum_{i=0}^{2k-1} (2k)^i$.
- There exists a $\varphi$-good valuation of $\mathcal{G}_i$ which refutes $\varphi$.

We show that $\alpha(\mathcal{G}_i)$, $1 \le i \le n$, is as required. To this end it suffices to prove for all descriptive $\mathcal{H}$ validating $\mathbf{CSM}_0$

$$\begin{array}{ll} & \mathcal{H}' \models \varphi, \text{ for all subframes } \mathcal{H}' \text{ of } \mathcal{H} \\ \Leftrightarrow & \mathcal{H} \models \alpha(\mathcal{G}_i), \text{ for all } 1 \le i \le n \ . \end{array}$$

Suppose $\mathcal{H}' \not\models \varphi$, for a subframe $\mathcal{H}'$ of $\mathcal{H}$. By Lemma 12 there exists $1 \le i \le n$ such that $\mathcal{G}_i$ is (isomorphic to) a subframe of the underlying Kripke frame of $\mathcal{H}'$. We conclude $\mathcal{H}' \not\models \alpha(\mathcal{G}_i)$, by Lemma 13. But the $\mathcal{H} \not\models \alpha(\mathcal{G}_i)$, as well. Conversely, suppose that $\mathcal{H} \not\models \alpha(\mathcal{G}_i)$, for an $1 \le i \le n$. Then, by Lemma 14 $\mathcal{H}' \not\models \varphi$, for a subframe $\mathcal{H}'$ of $\mathcal{H}$.

(i) Suppose that $\mathbf{CSM}_0 + \varphi$ is a subframe logic containing $\mathbf{CSM}_0$. Put $k = |\mathbf{Sub}\varphi|$. This time let $\mathcal{G}_i$, $1 \le i \le n$, be the collection of surrogate frames satisfying the following conditions:

- The cardinality of $\mathcal{G}_i$ is $\le \sum_{i=0}^{2k} (2k)^i$.
- There exists a $\varphi$-good valuation $\beta$ of $\mathcal{G}_i$ such that $\langle \mathcal{G}_i, \beta, r \rangle \not\models \varphi$, for the root $r$ of $\mathcal{G}_i$.

The proof that $\alpha(\mathcal{G}_i)$, $1 \le i \le n$, is as required is similar to the proof above and left to the reader. $\dashv$

# 5 Completeness

**Lemma 15** *Suppose that $\mathcal{H} = \langle W, R_1, R_2, P \rangle$ is descriptive and assume that $\mathcal{H} \models \mathbf{CSM}_0$. Suppose that $\mathcal{G} = \langle V, S_1, S_2 \rangle$ is a surrogate frame which is subframe of the underlying Kripke frame of $\mathcal{H}$ and suppose that $n \in \omega$. Then there exists a $\vec{A} \in Seq\mathcal{G}$ and a valuation $\beta$ in $\mathcal{H}$ such that*

- *$x \in \beta(p_x)$, for all $x \in V$.*
- *$\langle \mathcal{H}, \beta \rangle \models \delta(\mathcal{G}^n_{\vec{A}})$.*

**Proof.** Let $n \in \omega$. We first assume that the root of $\mathcal{G}$ is $R_2$-irreflexive. By Lemma 13 we can take a valuation $\gamma$ of $\mathcal{H}$ such that

- $x \in \gamma(p_x)$, for all $x \in V$.
- $\langle \mathcal{H}, \gamma \rangle \models \delta(\mathcal{G})$.

Call a subset $V_1$ of $V$ *closed* iff $x \in V_1$ and $xR_1y$ imply $y \in V_1$. Let $V_1$ be a closed subset of $V$ and put $\mathcal{F} = \langle V_1, S_1 \cap V_1 \times V_1, S_2 \cap V_1 \times V_1 \rangle$. Consider the following conditions on a valuation $\beta_{V_1}$ in $\mathcal{H}$. Here we denote by $\mathrm{Cl}(V_1)$ the set of $R_1$-clusters in $V_1$.

T1 $\gamma(p_x) \supseteq \beta_{V_1}(p_x)$, for all $x \in V$.

T2 $x \in \beta_{V_1}(p_x)$, for all $x \in V$.

T3 For all $C = \{j(C) : j < |C|\} \in \mathrm{Cl}(V_1)$ and $y \in C[n]$, $x \in C$:

$$\beta_{V_1}(p_y) \subseteq \gamma(p_x) \;\Leftrightarrow\; (\exists j < |C|)(\exists i \in \omega)(y = (i|C| + j, C) \wedge x = j(C)).$$

T4 There exists $\vec{A} = \langle A_x :\ x \in C,\ C \text{ a } R_1\text{-cluster in } \mathcal{G}\rangle \in \mathrm{Seq}\mathcal{G}$ such that for

$$\vec{A} \cap V_1 = \langle A_x \cap V_1 : x \in C,\ C \in \mathrm{Cl}(V_1)\rangle$$

we have

$$\langle \mathcal{H}, \beta_{V_1}\rangle \models \delta(\mathcal{F}^n_{(\vec{A}\cap V_1)})$$

and for all $x \in V - V_1$ and $y \in C[n]$, $C = \{j(C) : j < |C|\} \in \mathrm{Cl}(V_1)$ :

$$x \in \Diamond_2\beta_{V_1}(p_y) \Leftrightarrow (\exists j < |C|)(\exists i \in \omega)(y = (i|C| + j, C) \wedge x \in A_{j(C)}).$$

We show by induction on the cardinality of the closed sets $V_1$ that there always exists a valuation $\beta_{V_1}$ satisfying conditions [T1]-[T4]. The case $V_1 = \emptyset$ is trivial.

Suppose now that $\beta_{V_1}$ ($V_1$ possibly empty) is defined and $Q$ equals $\{x\}$ for an $R_1$-irreflexive point $x$ or $Q$ is a $R_1$-cluster. Assume that all proper $R_1$-successors of $Q$ are in $V_1$. Put $V_2 = V_1 \cup Q$. We are going to define a valuation $\beta_{V_2}$ satisfying the conditions [T1]-[T4] for $V_2$. Abbreviate

$$T(x, y) = (\exists C \in \mathrm{CL}(V_1))(\exists i \in \omega)(\exists j < |C|)(y = (i|C| + j, C) \wedge x \in A_{j(C)}).$$

and put for $x \in Q$

$$\begin{aligned} b_x \;=\; & \beta_{V_1}(p_x) \cap \\ & \bigcap\{\Diamond_i\beta_{V_1}(p_y) : xS^p_i y, y \in V_1, i \in \{1, 2\}\} \cap \\ & \bigcap\{\Diamond_2\beta_{V_1}(p_y) : T(x, y), y \in C, C \in \mathrm{Cl}(V_1)\} \cap \\ & \bigcap\{-\Diamond_2\beta_{V_1}(p_y) : \neg T(x, y), y \in C, C \in \mathrm{Cl}(V_1)\} \end{aligned}$$

We now distinguish two cases.

Case 1. $Q = \{x\}$, for an $R_1$-irreflexive $x$. In this case put

$$\beta_{V_2}(p_y) = \begin{cases} \beta_{V_1}(p_y) & : \; y \in (V - \{x\}) \cup \bigcup\{C[n] : C \in \mathrm{Cl}(V_1)\} \\ b_x & : \; y = x \end{cases}$$

Clearly $\beta_{V_2}$ satisfies conditions [T1]-[T4] for $V_2$.

Case 2. $Q$ is an $R_1$-cluster. Assume $Q = \{j(Q) : j < m\}$ and let $t \in \omega$.

**Claim 1.** There is a set of $R_1$-irreflexive points $x(i)$, $i < t+1$, such that $x(i+1)R_1x(i)$ for all $i < t$, and $QR_1x(t)$ and such that, for all $x \in Q$ and $k \leq t$:

$$x(k) \in b_x \quad \Leftrightarrow \quad (\exists i \in \omega)(\exists j < m)(k = im + j \wedge x = j(Q)).$$

*Proof of Claim 1.* The case $|Q| = 1$ is easy and left to the reader. Suppose now that $|Q| \geq 2$ and take two $x, y \in Q$. Then Claim 1 follows by induction if we can prove that there exist $R_1$-irreflexive $z_1 \in b_x$ and $z_2 \in b_y$ such that $z_2R_2z_1$. To this end take an $R_1$-irreflexive $z_1' \in b_x$ such that $QR_1z_1'$. Such a $z_1'$ exists by Lemma 11. We can choose $a_x \subseteq b_x$, $a_x \in P$, with $z_1' \in a_x \cap -\Diamond_1 a_x$. Put $a_y = b_y \cap \Diamond_1 a_x$. By Lemma 11 there exists an $R_1$-irreflexive $z_2 \in a_y$ with $QR_1z_2$. We find an $R_1$-irreflexive $z_1 \in a_x$ with $z_2R_1z_1$. These two points are as required and so Claim 1 is proved.

Assume now that $\{x(i) : i < t+1\}$ has the properties described in Claim 1. Notice that

$$R_2 \cap (g \times g) = \emptyset, \text{ for } g = Q \cup \{x(0), \ldots, x(t)\} \ .$$

This follows immediately from condition **Con** and $g \subseteq \bigcup\{\gamma(p_x) : x \in Q\}$. Hence we find a valuation $\gamma^*$ in $\mathcal{H}$ such that $\langle \mathcal{H}, \gamma^* \rangle \models \delta(\langle g, R_1 \cap g \times g, \emptyset \rangle)$ and

$$\gamma^*(p_{x(im+j)}) \subseteq b_{j(Q)}, \text{ for all } j \leq m-1 \text{ and } im+j \leq t,$$

$$x \in \gamma^*(p_x) \subseteq b_x, \text{ for all } x \in Q.$$

Certainly, by choosing $t \in \omega$ large enough we find a subsequence $\{x(n_0), \ldots, x(n_{n-1})\}$ of $\{x(0), \ldots, x(t)\}$ such that

$$x(n_{im+j}) \in b_{j(Q)}, \text{ for all } j < m \text{ and } im+j < n$$

and for each $y \in V - Q$ and $j(Q) \in Q$ with $yR_2j(Q)$

$$(\exists im+j < n)(y \in \Diamond_2\gamma^*(p_{x(n_{im+j})})) \quad \Rightarrow \quad (\forall im+j < n)(y \in \Diamond_2\gamma^*(p_{x(n_{im+j})}))$$

In other words, we find sets $B_x \subseteq \{y \in V : yS_2x\}$, $x \in Q$, and a valuation $\gamma^{**}$ in $\mathcal{H}$ such that $\langle \mathcal{H}, \gamma^{**} \rangle \models \delta(\langle Q, Q \times Q, \emptyset \rangle^n_\emptyset$ and

$$\gamma^{**}(p_{(im+j,Q)}) \subseteq b_{j(Q)}, \text{ for all } j < m \text{ and all } im+j < n,$$

$$x \in \gamma^{**}(p_x) \subseteq b_x, \text{ for all } x \in Q,$$

and for all $y \in V - Q$ and $j(Q) \in Q$ with $yR_2j(Q)$

$$(\forall im+j < n)[y \in \Diamond_2\gamma^{**}(p_{(im+j,Q)}) \Leftrightarrow y \in B_{j(Q)}].$$

We define

$$\vec{A}_{V_2} = \begin{cases} A_x & : \quad x \in C, \ C \in \mathrm{Cl}(V_1) \\ B_x & : \quad x \in Q \\ \emptyset & : \quad \text{otherwise} \end{cases}$$

and put

$$\beta_{V_2}(p_x) = \begin{cases} \beta_{V_1}(p_x) & : \quad x \in (V - Q) \cup \bigcup\{C[n] : C \in \mathrm{Cl}(V_1)\} \\ \gamma^{**}(p_x) & : \quad x \in Q \cup Q[n] \end{cases}$$

It is easy (but a bit tedious) to check that $\beta_{V_2}$ satisfies [T1]-[T4], where in [T4] we choose $\vec{A} = \vec{A}_{V_2}$.

For $V_1 = V$ the valuation $\beta_{V_1}$ is as required in the Lemma, by [T2] and [T4].

It remains to consider the case that the root $r$ of $\mathcal{G}$ is $R_2$-reflexive: Put $V_1 = V - \{r\}$ and $D = \{r\}$. Then we take a valuation $\beta_{V_1}$ in $\mathcal{H}$ satisfying [T1]-[T4] for $V_1$. Define $b_r$ as above. $r \in b_r$ is $R_2$-reflexive. One easily proves that there exists a chain $\{x(n-1)R_2 \ldots x(1)R_2x(0)\} \subseteq b_r$ of $R_1$-irreflexive points with $rR_2x(n-1)$. We find a valuation $\gamma^*$ in $\mathcal{H}$ with $r \in \gamma^*(p_r)$,

$$\langle \mathcal{H}, \gamma^* \rangle \models \delta(\langle D \cup D[n], S, S \rangle),$$

and $\gamma^*(p_x) \subseteq b_r$, for all $x \in D \cup D[n]$. Here

$$S = \{(r,r)\} \cup \{(r,(k,D)) : k < n\} \cup \{((j,D),(i,D)) : i < j < n\}.$$

Finally we put

$$\beta(p_x) = \begin{cases} \beta_{V_1}(p_x) & : \quad x \in V_1 \cup \bigcup\{C[n] : C \in \mathrm{Cl}(V_1)\} \\ \gamma^*(p_x) & : \quad x \in D \cup D[n] \end{cases}$$

$\beta$ is as required in the Lemma. ⊣

**Corollary 16** *Suppose that $\mathcal{H} = \langle W, R_1, R_2, P \rangle$ is descriptive and assume that $\mathcal{H} \models$ $\mathbf{CSM}_0$. Suppose that $\mathcal{G} = \langle V, S_1, S_2 \rangle$ is a surrogate frame with root $r$ which is a subframe of the underlying Kripke frame of $\mathcal{H}$. Then there exists an $\vec{A} \in Seq\mathcal{G}$ such that $\langle \mathcal{H}, r \rangle \not\models \alpha(\mathcal{G}^n_{\vec{A}})$, for all $n \in \omega$.*

**Proof.** Follows immediately from Lemma 15 by using the fact that Seq$\mathcal{G}$ is finite. ⊣

**Lemma 17** *Let $\mathcal{G}$ be a surrogate frame with root $r$, let $n \in \omega$ and $\vec{A} \in Seq\mathcal{G}$. For all $\varphi$:*

*(i) $\mathbf{G}[\mathcal{G}^\omega_{\vec{A}}] \not\models \varphi$ whenever there is a $\varphi$-good valuation $\beta$ in $\mathcal{G}^n_{\vec{A}}$ with $\langle \mathcal{G}^n_{\vec{A}}, \beta \rangle \not\models \varphi$.*

*(ii) $\langle \mathbf{G}[\mathcal{G}^\omega_{\vec{A}}], r \rangle \not\models \varphi$ whenever there is a $\varphi$-good valuation $\beta$ in $\mathcal{G}^n_{\vec{A}}$ with $\langle \mathcal{G}^n_{\vec{A}}, \beta, r \rangle \not\models \varphi$.*

**Proof.** (i)+(ii) Suppose $\langle \mathcal{G}^n_{\vec{A}}, \beta, z \rangle \not\models \varphi$, $\beta$ is $\varphi$-good. Let $\mathcal{G}^n_{\vec{A}} = \langle V, S_1, S_2 \rangle$ and $\mathbf{G}[\mathcal{G}^\omega_{\vec{A}}] = \langle W, R_1, R_2, P \rangle$. We define a valuation $\gamma$ in $\mathbf{G}[\mathcal{G}^\omega_{\vec{A}}]$ by putting

$$\gamma(p_x) = \begin{cases} \{x\} & : \quad x \in V \text{ is } R_1\text{-irreflexive} \\ \{j(C)\} \cup \{(i|C|+j, C) : i|C|+j \geq n\} & : \quad x = j(C),\ C \text{ cluster in } \mathcal{G} \end{cases}$$

It is easy to check now that

$$\bigcup\{\beta(p_x) : x \in V\} = W$$

and $\langle \mathbf{G}[\mathcal{G}^{\omega}_{\vec{A}}], \beta, r\rangle \not\models \alpha(\mathcal{G}^{n}_{\vec{A}})$. Hence, by the proof of Lemma 14, there exists a valuation $\gamma^*$ in $\mathbf{G}[\mathcal{G}^{\omega}_{\vec{A}}]$ such that $\langle \mathbf{G}[\mathcal{G}^{\omega}_{\vec{A}}], \gamma^*, z\rangle \not\models \varphi$. $\dashv$

**Lemma 18** *Let $\mathcal{G}$ be a surrogate frame with root $r$ and $\vec{A} \in Seq\mathcal{G}$. Put $t = 2k|C|$, where $C$ is a maximal cluster in $\mathcal{G}$ and $k$ is the cardinality of* $\mathbf{Sub}\varphi$.

*(i) For all $n \geq t$:*

$$\mathbf{G}[\mathcal{G}^{\omega}_{\vec{A}}] \not\models \varphi \Rightarrow \text{ there is a } \varphi\text{-good valuation } \beta \text{ in } \mathcal{G}^{n}_{\vec{A}} \text{ such that } \langle \mathcal{G}^{n}_{\vec{A}}, \beta\rangle \not\models \varphi.$$

*(ii) For all $n \geq t$:*

$$\langle \mathbf{G}[\mathcal{G}^{\omega}_{\vec{A}}], r\rangle \not\models \varphi \Rightarrow \text{ there is a } \varphi\text{-good valuation } \beta \text{ in } \mathcal{G}^{n}_{\vec{A}} \text{ such that } \langle \mathcal{G}^{n}_{\vec{A}}, \beta, r\rangle \not\models \varphi.$$

**Proof.** (i) + (ii) Let $n \geq t$. Suppose that $\langle \mathbf{G}[\mathcal{G}^{\omega}_{\vec{A}}], \gamma, x\rangle \not\models \varphi$. Following the proof of Lemma 12 we find a subframe $\mathcal{F}$ of $\mathcal{G}^{\omega}_{\vec{A}}$ which does not contain $R_1^p$-chains of length $\geq 2k$ such that the restriction $\gamma^*$ of $\gamma$ to $\mathcal{F}$ refutes $\varphi$ in $x$. But then we can certainly transform $\gamma^*$ into a $\varphi$-good valuation $\beta$ in $\mathcal{G}^{n}_{\vec{A}}$ which refutes $\varphi$ in $x$, whenever $x \in \mathcal{G}$ and in some $y \in C[n]$, whenever $x \in C[\omega]$, for a $R_1$-cluster $C$ in $\mathcal{G}$. $\dashv$

**Corollary 19** *(i) For all surrogate frames $\mathcal{G}$ with root $r$ and all $\vec{A} \in Seq\mathcal{G}$.*

- *The logic determined by $\mathbf{G}[\mathcal{G}^{\omega}_{\vec{A}}]$ is decidable.*
- *The logic determined by $\langle \mathbf{G}[\mathcal{G}^{\omega}_{\vec{A}}], r\rangle$ is decidable.*

**Proof.** Follows immediately from Lemma 17 and Lemma 18. $\dashv$

**Proof of Theorem 10** (i) Let $\Lambda$ be a normal subframe logic containing $\mathbf{CSM}_0$ and $\varphi \notin \Lambda$. Take a descriptive frame $\mathcal{H}$ validating $\Lambda$ such that $\mathcal{H} \not\models \varphi$. By Lemma 12 we find a normal surrogate frame $\mathcal{G}$ of cardinality $\leq \sum_{i=0}^{2k-1}(2k)^i$ which is a subframe of the underlying Kripke-frame of $\mathcal{H}$ such that there is a $\varphi$-good valuation $\beta^*$ in $\mathcal{G}$ which refutes $\varphi$. By Lemma 17, $\mathbf{G}[\mathcal{G}^{\omega}_{\vec{A}}] \not\models \varphi$, for all $\vec{A} \in \text{Seq}\mathcal{G}$. Hence it suffices to show that there exists $\vec{A} \in \text{Seq}\mathcal{G}$ such that $\mathbf{G}[\mathcal{G}^{\omega}_{\vec{A}}] \models \Lambda$. By Corollary 16 we can take an $\vec{A} \in \text{Seq}\mathcal{G}$ such that $\mathcal{H} \not\models \alpha(\mathcal{G}^{n}_{\vec{A}})$, for all $n \in \omega$. We show that this $\vec{A}$ is as required. To this end assume that $\mathbf{G}[\mathcal{G}^{\omega}_{\vec{A}}] \not\models \psi$. By Lemma 18 there exists $n \in \omega$ and a $\varphi$-good valuation $\beta$ in $\mathcal{G}^{n}_{\vec{A}}$ which refutes $\psi$ in $\mathcal{G}^{n}_{\vec{A}}$. But then a subframe of $\mathcal{H}$ refutes $\psi$, by Lemma 14. Hence $\psi \notin \Lambda$.

(ii) The proof of (ii) is similar and left to the reader. $\dashv$

**Proof of Theorem 6**

Follows immediately from Theorem 10 and Corollary 19. $\dashv$

The question arises whether all frames of the form $\mathbf{G}[\mathcal{G}^{\omega}_{\vec{A}}]$ are required in Theorem 10, or whether some subclass suffices to prove completeness. Call a (normal) surrogate frame $\mathcal{G} = \langle V, R_1, R_2 \rangle$ with root $r$ *selected* iff for all $x \neq r$ with $xR_1x$ there exists $y \in V$ such that $yR_2^p x$. Define a set $\mathrm{Seq}_0\mathcal{G}$ by putting $\vec{A} \in \mathrm{Seq}_0\mathcal{G}$ iff $\vec{A} \in \mathrm{Seq}\mathcal{G}$ and for all $x \neq r$ there exists $y \in V$ such that $yR_2^p x$ and $y \notin A_x$. Now the following holds:

**Theorem 20** *(i) Each normal subframe logic* $\Lambda$ *containing* $\mathbf{CSM}_0$ *is determined by a set of frames of the form* $\mathbf{G}[\mathcal{G}^{\omega}_{\vec{A}}]$*, where* $\mathcal{G}$ *is a selected normal surrogate frame and* $\vec{A} \in Seq_0\mathcal{G}$*.*

*(ii) Each quasi-normal subframe logic containing* $\mathbf{CSM}_0$ *is determined by a set of pointed frames of the form* $\langle \mathbf{G}[\mathcal{G}^{\omega}_{\vec{A}}], r \rangle$*, where* $\mathcal{G}$ *is a selected surrogate frame with root* $r$ *and* $\vec{A} \in Seq_0\mathcal{G}$*.*

For the proof notice that in the construction of $\mathcal{G}$ in the proof of Lemma 12 a $R_1$-reflexive point $x$ was selected only in the case that (i) there exists $\psi \in \mathbf{Sub}\varphi$ such that $x \in \max_{R_2}(\beta(\psi))$ and (ii) there exists an already selected $y \in \mathcal{G}$ such that $yR_2x$. Hence all frames $\mathcal{G}$ constructed in Lemma 12 are *selected*. Moreover, we may assume that we select an $R_1$-irreflexive point $x$ with $yR_2x$ and $x \in \max_{R_2}(\beta(\psi))$ whenever this is possible. In other words, we may assume that an $R_1$-reflexive point $x$ was selected in the construction of $\mathcal{G}$ iff

- there exists $\psi \in \mathbf{Sub}\varphi$ such that $x \in \max_{R_2}(\beta(\psi))$
- and there exists an already selected $y \in \mathcal{G}$ such that $yR_2x$ and such that there does not exist an $R_1$-irreflexive point $z$ with $yR_2z$ and $z \in \max_{R_2}(\beta(\psi))$.

The construction of $\mathcal{G}^n_{\vec{A}}$ in Lemma 15 shows that for such a subframe $\mathcal{G}$ of the underlying Kripke frame of $\mathcal{H}$ we have $\vec{A} \in \mathrm{Seq}_0\mathcal{G}$. A rather tedious proof shows now that the class of frames

$$\mathbf{M} = \{\mathbf{G}[\mathcal{G}^{\omega}_{\vec{A}}] : \ \mathcal{G} \text{ a normal selected surrogate frame, } \vec{A} \in \mathrm{Seq}_0\mathcal{G}\}$$

is minimal, i.e., for each proper subclass $\mathbf{N}$ of $\mathbf{M}$ closed under isomorphic images there exists a normal subframe logic $\Lambda$ which is not determined by a subclass of $\mathbf{N}$. Hence no further simplification of the completeness theorem is possible. For specific systems, however, we easily derive completeness with respect to smaller classes. As an illustration we prove that the logic $\mathbf{CSM}_0$ has the finite model property: Suppose that $\varphi \notin \mathbf{CSM}_0$. There exists a normal surrogate frame $\mathcal{G}$ with a $\varphi$-good valuation $\beta$ such that $\langle \mathcal{G}, \beta \rangle \not\models \varphi$. Replace all $R_1$-clusters in $\mathcal{G}$ by sets of $R_1$-irreflexive points of the same cardinality and denote the result by $\mathcal{G}'$. Then $\mathcal{G}'$ validates $\mathbf{CSM}_0$ but $\langle \mathcal{G}', \beta \rangle$ still refutes $\varphi$.

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