Intuitionistic Modal Logics as Fragments of Classical Bimodal Logics

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In memory of Helena Rasiowa

Abstract

Gödel's translation of intuitionistic formulas into modal ones provides the well-known embedding of intermediate logics into extensions of Lewis' system **S4**, which reflects and sometimes preserves such properties as decidability, Kripke completeness, the finite model property. In this paper we establish a similar relationship between intuitionistic modal logics and classical bimodal logics. We also obtain some general results on the finite model property of intuitionistic modal logics first by proving them for bimodal logics and then using the preservation theorem.

The aim of this paper is to show how well known results and technique developed in the field of intermediate and classical (mono- and poly-) modal logics can be used for studying various intuitionistic modal systems.

The current state of knowledge in intuitionistic modal logic resembles (to some extent) that in classical modal logic about a quarter of a century ago, when the discipline was just an extensive collection of individual systems and the best method of proving decidability was the art of filtration. The reason why intuitionistic modal logic lags far behind its classical counterpart is quite apparent. Intuitionistic modal logics are much more closely related to classical bimodal logics than to usual monomodal ones, and only few recent years have brought fairly general results in polymodal logic (e.g. [9], [13], [20]). This explains also the strange fact that the embedding of intuitionistic modal logics into classical bimodal ones (via the evident generalization of Gödel's translation

*The work of the second author was supported by the Alexander von Humboldt Foundation.

of intuitionistic logic into **S4**), mentioned by Fisher Servi [10] and Shehtman [18] at the end of the 70th, has not found really worthy applications.

In a series of two papers we are going to investigate into features of this embedding and use them to transfer rather general results on the finite model property (and so decidability) from classical bimodal logics to their intuitionistic fragments.

Here we consider extensions of **IntK**, intuitionistic propositional logic with the necessity operator $\Box$ of the minimal classical modal system **K**, introduced and investigated in [3], [19]. The possibility operator $\Diamond$ is defined via $\Box$ and $\neg$ as in classical modal logic (although this does not mean that $\Box$ and $\Diamond$ are dual as in **K**; see [11]). In the next paper [21] we extend a part of the theory developed here to systems with weaker connections between $\Box$ and $\Diamond$, e.g. to those in [15], [19], [1].

In Section 1 we define relational and algebraic semantics for intuitionistic modal logics and combine elements of duality theory for modal and intermediate logics to obtain basic duality results for intuitionistic modal algebras and frames. Taking for granted the reader's acquaintance with classical duality theory (consult, for example, [12] or [5]), we omit almost all proofs. In Section 2 we show that every normal extension of **IntK** can be embedded by the (extended) Gödel translation into bimodal logics forming an interval $[\tau_0 L, \sigma L]$ and that the embedding reflects decidability, Kripke completeness, the finite model property and tabularity. (In [21] we prove an unexpected analog of Blok-Esakia's theorem (on an isomorphism between the lattices of intermediate logics and extensions of the Grzegorczyk system **Grz**; see [2], [6]) establishing an isomorphism between the lattices of extensions of **IntK** and σ**IntK**). Then, in Section 3, using results and methods of [8], [13], [23] and [25], we prove some general theorems on the finite model property (FMP) of bimodal logics and thereby of their intuitionistic counterparts. Namely, first we show that the extensions of **IntK**, **IntT** and **IntD** with a set Γ of intuitionistic formulas preserve such properties of the intermediate logic **Int** $+ \Gamma$ as FMP, decidability and Kripke completeness. And second, roughly we extend to intuitionistic modal logics the results of [8], [23] and [25] on the FMP of (mono) modal and intermediate subframe and cofinal subframe logics.

It is to be emphasized once again that here we do not develop a special completeness theory for intuitionistic modal logics. The pathos of the paper is that classical modal logic has enough power to throw light on its more exotic neighbors.

1 Semantics

The basic intuitionistic modal system **IntK** we deal with in this paper is formulated in the language $\mathcal{L}\Box$, which has the connectives $\rightarrow$, $\wedge$, $\vee$, $\bot$ and $\Box$ ($\neg\varphi$ is defined as $\varphi \rightarrow \bot$ and $\Diamond\varphi$ as $\neg\Box\neg\varphi$). Its axioms are the standard axioms of propositional intuitionistic logic **Int**, the modal axiom of the minimal normal modal logic **K**, i.e., $\Box(p \to q) \to (\Box p \to \Box q)$, and the inference rules are substitution, modus ponens and necessitation. By an *intuitionistic* (or better *superintuitionistic*) *modal logic* (*IM-logic*, for short) we mean any set of $\mathcal{L}\Box$-formulas containing the axioms of **IntK** and closed under the rules mentioned above. For a set Γ of $\mathcal{L}\Box$-formulas and an IM-logic L, we denote by $L \oplus \Gamma$ the smallest IM-logic to contain Γ and L.

An *IM-frame* is a structure $\mathfrak{F} = \langle W, \leq, R, P\rangle$, where W is a non-empty set, $\leq$ a partial order and R a binary relation on W such that

$$\leq \circ R = R \circ \leq \; = R, \tag{1}$$

and P is a set of cones (= upward closed sets) in W with respect to $\leq$ containing $\emptyset$ and closed under $\cap$, $\cup$ and the operations $\supset$ and $\Box$ defined by

$$X \supset Y = \{x \in W : \forall y \in W \; (x \leq y \;\&\; y \in X \Rightarrow y \in Y)\},$$
$$\Box X = \{x \in W : \forall y \in W \; (xRy \Rightarrow y \in X\} \tag{2}$$

(which interpret the connectives $\wedge$, $\vee$, $\to$ and $\Box$, respectively). It follows from (1) that $\Box X$ is a cone whenever X is a cone. A *valuation* in $\mathfrak{F}$ is a map $\mathfrak{V}$ from the set of $\mathcal{L}\Box$'s variables in P. As usual $\mathfrak{V}$ is extended to the set of $\mathcal{L}\Box$-formulas ($\mathfrak{V}(\varphi \to \psi) = \mathfrak{V}(\varphi) \supset \mathfrak{V}(\psi)$, $\mathfrak{V}(\Box\varphi) = \Box\mathfrak{V}(\varphi)$, etc.), and the *truth-relation* $\models$ in the model $\langle \mathfrak{F}, \mathfrak{V}\rangle$ is defined by $x \models \varphi$ iff $x \in \mathfrak{V}(\varphi)$.

It is easily checked that every IM-frame $\mathfrak{F}$ validates **IntK** ($\mathfrak{F} \models$ **IntK**, in symbols), i.e., under any valuation $x \models \varphi$ for every point x in $\mathfrak{F}$ and every $\varphi \in$ **IntK**. And using the standard canonical model technique one can prove that every IM-logic L is *characterized* by a suitable class $\mathcal{C}$ of IM-frames in the sense that $\varphi \in L$ iff $\mathfrak{F} \models \varphi$ for all $\mathfrak{F} \in \mathcal{C}$ (all essential details can be found in [3]).

An IM-frame $\mathfrak{F} = \langle W, \leq, R, P\rangle$ is called a *Kripke IM-frame* if P consists of all cones in W. The underlying Kripke IM-frame of an IM-frame $\mathfrak{F}$ is denoted by $\kappa\mathfrak{F}$. An IM-logic is *Kripke complete* if it is characterized by a class of Kripke IM-frames.

An *IM-algebra* is a structure $\mathfrak{A} = \langle A, \to, \wedge, \vee, \top, \Box\rangle$ such that $\langle A, \to, \wedge, \vee, \top\rangle$ is a pseudo-Boolean (= Heyting) algebra with unit element $\top$ and, for every $a, b \in A$,

$$\Box\top = \top, \quad \Box(a \to b) \to (\Box a \to \Box b) = \top.$$

Each IM-logic corresponds to the variety of IM-algebras in which all its formulas are identically equal to $\top$. With every IM-algebra $\mathfrak{A}$ we can associate an IM-frame $\mathfrak{A}_+ = \langle W, \leq, R, P\rangle$, called the *Stone representation* of $\mathfrak{A}$, by taking (cf. [12], [5]) W to be the set of prime filters in $\mathfrak{A}$ and, for $x, y \in W$,

$$x \leq y \text{ iff } x \subseteq y,$$

$$xRy \text{ iff } \forall a \in A\ (\Box a \in x \Rightarrow a \in y),$$

$$P(a) = \{x \in W:\ a \in x\},$$

$$P = \{P(a):\ a \in A\}.$$

Conversely, each IM-frame $\mathfrak{F} = \langle W, \leq, R, P\rangle$ gives rise to the IM-algebra $\mathfrak{F}^+ = \langle P, \supset, \cap, \cup, W, \Box\rangle$. It is a matter of routine (consult e.g. [12], [5]) to show that, for every IM-algebra $\mathfrak{A}$, $\mathfrak{A} \simeq (\mathfrak{A}_+)^+$.

As in [12], we call an IM-frame $\mathfrak{F}$ *descriptive* if $\mathfrak{F} \simeq (\mathfrak{F}^+)_+$. In the standard way one can prove the following internal characterization of descriptive IM-frames.

Proposition 1 *An IM-frame $\mathfrak{F} = \langle W, \leq, R, P\rangle$ is descriptive iff it is tight$_\leq$, i.e.,*

$$x \leq y \textit{ iff } \forall X \in P\ (x \in X \Rightarrow y \in X),$$

tight$_R$, i.e.,

$$xRy \textit{ iff } \forall X \in P\ (x \in \Box X \Rightarrow y \in X),$$

and compact, i.e., for every $\mathcal{X} \subseteq P$ and $\mathcal{Y} \subseteq \{W - X:\ X \in P\}$, if $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property then $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$.

An IM-logic L is said to be *d-persistent* if, for every descriptive frame $\mathfrak{F}$, $\kappa\mathfrak{F} \models L$ whenever $\mathfrak{F} \models L$.

Following the well known path (cf. [12], [5]) one can develop further duality theory between IM-algebras and IM-frames; we do not need that much here.

We are going to embed IM-logics into classical bimodal normal logics in the language $\mathcal{L}\Box_I\Box_M$ of propositional modal logic with two necessity operators $\Box_I$ and $\Box_M$. Here our basic bimodal system is $\mathbf{S4} \otimes \mathbf{K}$, which is obtained by putting together the axioms of Lewis' **S4** with the necessity operator $\Box_I$, that of **K** with $\Box_M$ and taking the closure under substitution, modus ponens and necessitation with respect to both $\Box_I$ and $\Box_M$. By a *bimodal logic* (*BM-logic*, for short) we mean any extension of $\mathbf{S4} \otimes \mathbf{K}$ in the language $\mathcal{L}\Box_I\Box_M$ closed under all those rules. $L \oplus \Gamma$ is the smallest BM-logic containing a BM-logic L and a set of $\mathcal{L}\Box_I\Box_M$-formulas Γ. For normal monomodal logics $L_1 \supseteq \mathbf{S4}$ and L_2, formulated in the languages with $\Box_I$ and $\Box_M$, respectively, the *fusion* $L_1 \otimes L_2$ is the smallest BM-logic to contain $L_1 \cup L_2$.

BM-logics are interpreted by *BM-frames* which are structures of the form $\mathfrak{F} = \langle W, R_I, R_M, P\rangle$, where W is a non-empty set, R_I a quasi-order and R_M an arbitrary binary relation on W, $P \subseteq 2^W$ contains $\emptyset$ and is closed under the Boolean operations and the operations $\Box_I$ and $\Box_M$ defined by (2) with $\Box$, R replaced by $\Box_I$, R_I and $\Box_M$, R_M, respectively. $\mathfrak{F}$ is a *Kripke BM-frame* if $P = 2^W$. It is well known that every BM-logic is characterized by a class of BM-frames. The other notions and results from duality theory for polymodal logics used in this paper are similar to those in the monomodal case; the reader can find them in [12] or [5].

2 Embedding

Now we define inductively a translation t of $\mathcal{L}\Box$ into $\mathcal{L}\Box_I\Box_M$ by taking

$$
\begin{aligned}
t(p) &= \Box_I p, \ p \text{ a variable},\\
t(\bot) &= \Box_I \bot,\\
t(\varphi \to \psi) &= \Box_I(t(\varphi) \to t(\psi)),\\
t(\varphi \wedge \psi) &= \Box_I(t(\varphi) \wedge t(\psi)),\\
t(\varphi \vee \psi) &= \Box_I(t(\varphi) \vee t(\psi)),\\
t(\Box\varphi) &= \Box_I\Box_M t(\varphi).
\end{aligned}
$$

The restriction of t to the language of intuitionistic logic is clearly nothing else but Gödel's embedding of **Int** into **S4**. We are going to show that t embeds IM-logics into BM-logics. With this in mind we define first two operators transforming IM-frames into BM-frames and back.

Given an IM-frame $\mathfrak{F} = \langle W, \leq, R, P\rangle$, construct a bimodal frame $\sigma\mathfrak{F} = \langle W, R_I, R_M, \sigma P\rangle$ simply by taking $R_I = \leq$, $R_M = R$ and σP to be the Boolean closure of P.

Lemma 2 *If $\mathfrak{F}$ is an IM-frame then $\sigma\mathfrak{F}$ is a BM-frame.*

Proof. It suffices to show that σP is closed under $\Box_I$ and $\Box_M$. That σP is closed under $\Box_I$ follows from [14]. Suppose that $X \in \sigma P$. Then, by Theorem II.2.2 in [16], there are sets $Y_i, Z_i \in P$, for $i = 1, \dots, n$, such that

$$X = \bigcap_{i=1}^{n}(-Y_i \cup Z_i).$$

It follows from the definitions of $\Box_I$ and $\supset$ that

$$\Box_I(-Y_i \cup Z_i) = Y_i \supset Z_i \in P.$$

And since, by (1), $R_M \circ R_I = R_M$ and $\Box_I$ distributes over intersections, we have

$$\Box_M X = \Box_M \bigcap_{i=1}^{n} \Box_I(-Y_i \cup Z_i) = \Box_M \bigcap_{i=1}^{n}(Y_i \supset Z_i),$$

with the latter set being in P because P is closed under $\Box$. $\dashv$

Now let $\mathfrak{F} = \langle W, R_I, R_M, P\rangle$ be a BM-frame. We construct an IM-frame denoted by $\rho\mathfrak{F}$ in three steps. First define a BM-frame $\mathfrak{F}^* = \langle W, R_I, R_M^*, P\rangle$ by putting, for every $x, y \in W$,

$$xR_M^* y \text{ iff } xR_I \circ R_M \circ R_I y.$$

Since R_I is a quasi-order, we clearly have $R_M \subseteq R_M^*$.

Lemma 3 *Suppose $\mathfrak{F}$ is a BM-frame. Then*

(i) *$\mathfrak{F}^*$ is also a BM-frame;*

(ii) *the equality*

$$R_I \circ R_M^* = R_M^* \circ R_I = R_M^* \tag{3}$$

holds in $\mathfrak{F}^*$ or, which is equivalent, $\mathfrak{F}^*$ validates the formula

$$Mix = (\Box_I \Box_M p \leftrightarrow \Box_M p) \wedge (\Box_M \Box_I p \leftrightarrow \Box_M p);$$

(iii) *for every $\mathcal{L}\Box$-formula φ, $\mathfrak{F}^* \models t(\varphi)$ iff $\mathfrak{F} \models t(\varphi)$;*

(iv) *if $\mathfrak{F}$ is descriptive then $\mathfrak{F}^*$ is descriptive too.*

Proof. The proof of the first three items is straightforward ((iii) uses the fact that each formula $t(\psi)$ begins with $\Box_I$).

(iv) Clearly, it is enough to show that $\mathfrak{F}^*$ is tight with respect to R_M^*, i.e., that, for every $x, y \in W$,

$$xR_M^* y \text{ iff } \forall X \in P(y \in X \Rightarrow x \in \Diamond_M^* X),$$

where $\Diamond_M^* X = \{u \in W : \exists v \in X \ uR_M^* v\}$, or, which is equivalent,

$$\begin{array}{rl} xR_I \circ R_M \circ R_I y & \text{iff} \quad \forall X \in P(y \in X \Rightarrow x \in \Diamond_I \Diamond_M \Diamond_I X) \\ & \text{iff} \quad x \in \bigcap \{\Diamond_I \Diamond_M \Diamond_I X : \ y \in X \in P\}. \end{array}$$

Since $\mathfrak{F}$ is compact and tight, the last inclusion can be transformed, by Esakia's Lemma (see e.g. [17]), to

$$x \in \Diamond_I \Diamond_M \Diamond_I \bigcap \{X : \ y \in X \in P\}$$

which, by differentiatedness, is nothing else but $x \in \Diamond_I \Diamond_M \Diamond_I \{y\}$, i.e., $xR_I \circ R_M \circ R_I y$. $\dashv$

Suppose now that a BM-frame $\mathfrak{F} = \langle W, R_I, R_M, P\rangle$ validates Mix. Define an equivalence relation $\sim$ on W by taking $x \sim y$ iff x and y belong to the same R_I-cluster in $\mathfrak{F}$ (i.e., $xR_I y$ and $yR_I x$) and let $\lfloor x \rfloor = x/_\sim$ and $\lfloor X \rfloor = \{\lfloor x \rfloor : \ x \in X\}$, for any $x \in W$ and $X \subseteq W$. Put

$$\lfloor x \rfloor \lfloor R_I \rfloor \lfloor y \rfloor \text{ iff } xR_I y,$$

$$\lfloor x \rfloor \lfloor R_M \rfloor \lfloor y \rfloor \text{ iff } xR_M y,$$

$$\lfloor P \rfloor = \{\lfloor X \rfloor : \ \bigcup \lfloor X \rfloor \in P\}.$$

(Since R_I is transitive and since, by Mix, $xR_M y$ iff $zR_M y$, for every x and z belonging to the same R_I-cluster, the definition of $\lfloor R_I \rfloor$ and $\lfloor R_M \rfloor$ does not depend on the choice of representatives in the classes $\lfloor x \rfloor$ and $\lfloor y \rfloor$.) The structure $\lfloor \mathfrak{F} \rfloor = \langle \lfloor W \rfloor, \lfloor R_I \rfloor, \lfloor R_M \rfloor, \lfloor P \rfloor \rangle$ is called the *skeleton* of $\mathfrak{F}$. It is easy to see that if $\mathfrak{F}$ validates Mix and R_I is a partial order then $\mathfrak{F} \simeq \lfloor \mathfrak{F}^* \rfloor$.

Lemma 4 *Suppose a BM-frame $\mathfrak{F}$ validates Mix. Then*

(i) *the map $x \mapsto \lfloor x \rfloor$ is a p-morphism from $\mathfrak{F}$ onto $\lfloor \mathfrak{F} \rfloor$ (or, in other terms, $\lfloor \mathfrak{F} \rfloor^+$ is a subalgebra of $\mathfrak{F}^+$) and so $\lfloor \mathfrak{F} \rfloor$ is a BM-frame;*

(ii) *$\lfloor R_I \rfloor$ is a partial order on $\lfloor W \rfloor$ and*

$$\lfloor R_I \rfloor \circ \lfloor R_M \rfloor = \lfloor R_M \rfloor \circ \lfloor R_I \rfloor = \lfloor R_M \rfloor ;$$

(iii) *for every $\mathcal{L}\Box$-formula φ, $\mathfrak{F} \models t(\varphi)$ iff $\lfloor \mathfrak{F} \rfloor \models t(\varphi)$.*

Proof. Straightforward. ⊣

Now, given an arbitrary BM-frame $\mathfrak{F} = \langle W, R_I, R_M, P\rangle$, we form the frame $\lfloor \mathfrak{F}^* \rfloor = \langle \lfloor W \rfloor, \lfloor R_I \rfloor, \lfloor R_M^* \rfloor, \lfloor P \rfloor\rangle$ and then define $\rho\mathfrak{F} = \langle \lfloor W \rfloor, \leq, R, \rho P\rangle$ by taking $\leq\, = \lfloor R_I \rfloor$, $R = \lfloor R_M^* \rfloor$ and $\rho P = \{\Box_I X : X \in \lfloor P \rfloor\}$. Since, for every X and Y, $\Box_I X \supset \Box_I Y = \Box_I(-\Box_I X \cup \Box_I Y)$, it should be clear that $\rho\mathfrak{F}$ is an IM-frame.

Lemma 5 *For every $\mathcal{L}\Box$-formula φ and every BM-frame $\mathfrak{F}$,*

$$\mathfrak{F} \models t(\varphi) \text{ iff } \rho\mathfrak{F} \models \varphi.$$

Proof. By induction on the construction of φ using Lemmas 3 and 4. ⊣

Lemma 6 *$\mathfrak{F} \simeq \rho\sigma\mathfrak{F}$, for every IM-frame $\mathfrak{F}$.*

Proof. It suffices to observe in the proof of Lemma 2 that $\Box_I X \in P$ for every $X \in \sigma P$. ⊣

The operators σ and ρ defined above are just natural generalizations of the similar operators connecting intuitionistic and **S4**-frames. For details consult [5], where it is proved, in particular, that these operators preserve descriptiveness. We extend this result to

Proposition 7 (i) *If $\mathfrak{F}$ is a descriptive BM-frame then $\rho\mathfrak{F}$ is a descriptive IM-frame.*

(ii) *If $\mathfrak{F}$ is a descriptive IM-frame then $\sigma\mathfrak{F}$ is a descriptive BM-frame.*

Proof. If we forget about R_M and R, then ρ and σ are the above mentioned operators on general frames for **S4** and **Int**, respectively, preserving descriptiveness. So (ii) is clear and in (i) it remains to show that $\rho\mathfrak{F}$ is tight$_R$. Suppose otherwise. Then there are $\lfloor x \rfloor, \lfloor y \rfloor \in \lfloor W \rfloor$ such that, for all $\lfloor X \rfloor \in \rho P$, $\lfloor x \rfloor \in \Box \lfloor X \rfloor \Rightarrow \lfloor y \rfloor \in \lfloor X \rfloor$ but not $\lfloor x \rfloor R \lfloor y \rfloor$. By Lemma 3, we may assume that $\mathfrak{F}$ validates Mix, i.e., $R_M = R_M^*$. Then $xR_M y$ does not hold and so, since $\mathfrak{F}$ is tight, there is $X \in P$ such that $x \in \Box_M X$ and $y \notin X$. By Mix, $x \in \Box_M \Box_I X$ and, since R_I is a quasi-order, $y \notin \Box_I X$. Clearly we also have $\lfloor \Box_I X \rfloor \in \rho P$. But this leads us to a contradiction, because $\lfloor x \rfloor \in \Box \lfloor \Box_I X \rfloor$ and $\lfloor y \rfloor \notin \lfloor \Box_I X \rfloor$. ⊣

Say that an IM-logic L is *embedded* in a BM-logic M by t if, for every $\mathcal{L}\Box$-formula φ,

$$\varphi \in L \text{ iff } t(\varphi) \in M.$$

In this case we call M a *BM-companion* of L and L the *IM-fragment* of M. If M is a bimodal logic then one can readily check that the set

$$\rho M = \{\varphi \in \mathcal{L}\Box : \ t(\varphi) \in M\}$$

is an IM-logic and so ρM is the IM-fragment of M. It is not hard to see also that ρ is a homomorphism from the lattice of BM-logics onto the lattice of IM-logics.

As an immediate consequence of the definition and Lemma 5 we obtain

Proposition 8 *If a BM-logic M is characterized by a class $\mathcal{C}$ of BM-frames then ρM is characterized by the class $\rho\mathcal{C} = \{\rho\mathfrak{F} : \ \mathfrak{F} \in \mathcal{C}\}$ of IM-frames.*

Theorem 9 *Each IM-logic $L = \mathbf{IntK} \oplus \Gamma$ is embeddable by t in any logic M in the interval*

$$[(\mathbf{S4} \otimes \mathbf{K}) \oplus t(\Gamma), (\mathbf{Grz} \otimes \mathbf{K}) \oplus t(\Gamma) \oplus Mix],$$

where $\mathbf{Grz} = \mathbf{S4} \oplus \Box_I(\Box_I(p \to \Box_I p) \to p) \to p$.

Proof. We show that $L = \rho M$. Suppose $\varphi \notin L$. Then there is an IM-frame $\mathfrak{F}$ for L refuting φ. By Lemmas 5 and 6, $\sigma\mathfrak{F} \not\models t(\varphi)$. By the same reason we have $\sigma\mathfrak{F} \models t(\Gamma)$. As is well known (cf. [5]), $\sigma\mathfrak{F}$ is a frame for the Grzegorczyk formula in the language with $\Box_I$. And that $\sigma\mathfrak{F}$ validates Mix follows from the definition. Thus, we obtain $\sigma\mathfrak{F} \models M$ and $\sigma\mathfrak{F} \not\models t(\varphi)$, from which $t(\varphi) \notin M$ and so $\varphi \notin \rho M$.

Conversely, if $\varphi \notin \rho M$ then $t(\varphi) \notin M$ and so there is a BM-frame $\mathfrak{F}$ for M refuting $t(\varphi)$. By Lemma 5, $\rho\mathfrak{F} \not\models \varphi$ and $\rho\mathfrak{F} \models \Gamma$. Hence $\varphi \notin L$. $\dashv$

Example 10 The BM-logic

$$(\mathbf{S4} \otimes \mathbf{K4}) \oplus Mix = (\mathbf{S4} \otimes \mathbf{K}) \oplus \Box_M p \to \Box_M \Box_M p \oplus Mix$$

is a BM-companion of the IM-logic

$$\mathbf{IntK4} = \mathbf{IntK} \oplus \Box p \to \Box\Box p.$$

Indeed, by Mix, we have

$$t(\Box p \to \Box\Box p) \leftrightarrow \Box_I(\Box_M p \to \Box_M \Box_M p) \in (\mathbf{S4} \otimes \mathbf{K}) \oplus Mix.$$

It will follow also from the proof of Theorem 13 that

$$(\mathbf{S4} \otimes \mathbf{S4}) \oplus Mix = (\mathbf{S4} \otimes \mathbf{S4}) \oplus \Box_M p \to \Box_I p$$

is a BM-companion of $\mathbf{IntS4} = \mathbf{IntK4} \oplus \Box p \to p$.

It is worth noting that every BM-companion M of an IM-logic L can be reduced, in a sense, to a BM-companion of L containing Mix. Say that a BM-logic M' is a *Mix-reduct* of a BM-logic M if $Mix \in M'$ and, for every formula φ, $\varphi \in M'$ iff $r(\varphi) \in M$, where r replaces each occurrence of $\Box_M$ in φ with $\Box_I \Box_M \Box_I$. Then, by Lemma 3, for each BM-companion M of an IM-logic L, there exists a Mix-reduct M' of M such that $\rho M' = L$ (if M is characterized by a frame $\mathfrak{F}$ then M' can be defined as the logic of $\mathfrak{F}^*$). However, it would be of interest to clarify the structure of the whole set $\rho^{-1}L$ for an arbitrary IM-logic L.

As for the BM-companions of IM-logics which contain Mix, in our forthcoming paper [21] we get a correspondence similar to that between intermediate logics and their modal companions above **S4** (see [4]). Namely, we show that

(a) the logic $\sigma L = (\mathbf{Grz} \otimes \mathbf{K}) \oplus t(\Gamma) \oplus Mix$ is the greatest BM-companion of $L = \mathbf{IntK} \oplus \Gamma$ containing Mix (the logic $\tau L = (\mathbf{S4} \otimes \mathbf{K}) \oplus t(\Gamma) \oplus Mix$ is clearly the smallest one),

(b) σL is characterized by the class $\sigma \mathcal{C} = \{\sigma \mathfrak{F} : \mathfrak{F} \in \mathcal{C}\}$ whenever L is characterized by a class $\mathcal{C}$ of IM-frames and

(c) the map σ is an isomorphism from the lattice of IM-logics onto the lattice of extensions of $(\mathbf{Grz} \otimes \mathbf{K}) \oplus Mix$.

In the next section, to prove our completeness results we require the following preservation theorems. (The proofs of some claims below take advantage of the facts (a) and (b) mentioned above. They are not used in Section 3; we present them here only for the integrity of the picture.)

Theorem 11 *The map ρ preserves decidability, Kripke completeness, FMP and tabularity. The map σ preserves FMP and tabularity.*

Proof. That ρ preserves decidability follows directly from the definition of ρ and the rest from Proposition 8, (b) and the fact that $\rho\mathfrak{F}$ is a Kripke (finite) IM-frame whenever $\mathfrak{F}$ is a Kripke (or finite) BM-frame and $\sigma\mathfrak{F}$ is finite whenever $\mathfrak{F}$ is finite. ⊣

Theorem 12 (i) *If a BM-logic M containing Mix is d-persistent then ρM is also d-persistent.*

(ii) *If an IM-logic L is d-persistent then τL is also d-persistent.*

Proof. (i) Let $\mathfrak{F}$ be a descriptive frame for ρM. Then, by Proposition 7 and (b), $\sigma\mathfrak{F}$ is a descriptive frame for $\sigma\rho M$ and so, by (a), for M. Since M is d-persistent, we have $\kappa\sigma\mathfrak{F} \models M$ and hence $\rho\kappa\sigma\mathfrak{F} \models \rho M$. It remains to observe that $\rho\kappa\sigma\mathfrak{F} \simeq \kappa\mathfrak{F}$.

(ii) Let $\mathfrak{F}$ be a descriptive frame for τL. By Propositions 7 and 8, $\rho\mathfrak{F}$ is a descriptive frame for $\rho\tau L = L$, from which $\kappa\rho\mathfrak{F} \models L$. Clearly, $\kappa\rho\mathfrak{F} \simeq \rho\kappa\mathfrak{F}$, whence, by Lemma 5, we have $\kappa\mathfrak{F} \models \tau L$. ⊣

3 Completeness

We obtain our completeness, in particular, FMP results in two steps. First we lift them from intermediate and monomodal logics to BM-logics using the preservation theorems from [13] and [22] and the method of maximal points from [8]. And then we apply Theorem 11 to transfer these results from BM-logics to their IM-fragments.

Theorem 13 *Suppose that an intermediate logic* $\mathbf{Int} + \Gamma$ *(in the language* $\mathcal{L}$ *of intuitionistic logic*[1]*) has one of the properties*

- *the finite model property;*
- *decidability and Kripke completeness;*
- *Kripke completeness.*

Then the IM-logics $\mathbf{IntK} \oplus \Gamma$, $\mathbf{IntK} \oplus \Gamma \oplus \Box p \to p$ *and* $\mathbf{IntK} \oplus \Gamma \oplus \Diamond\top$ *also have the same property.*

Proof. By Theorem 11, it suffices to show that there is a BM-companion of each of these systems satisfying the corresponding property. Notice that

$$\rho((\mathbf{S4} \oplus t(\Gamma)) \otimes \mathbf{K}) = \mathbf{IntK} \oplus \Gamma,$$
$$\rho((\mathbf{S4} \oplus t(\Gamma)) \otimes (\mathbf{K} \oplus \Box_M p \to p)) = \mathbf{IntK} \oplus \Gamma \oplus \Box p \to p,$$
$$\rho((\mathbf{S4} \oplus t(\Gamma)) \otimes (\mathbf{K} \oplus \Diamond_M \top)) = \mathbf{IntK} \oplus \Gamma \oplus \Diamond\top.$$

The first equation holds because $(\mathbf{S4} \oplus t(\Gamma)) \otimes \mathbf{K} = (\mathbf{S4} \otimes \mathbf{K}) \oplus t(\Gamma)$. The second one is a consequence of Theorem 9 and the inclusions

$$(\mathbf{S4} \otimes \mathbf{K}) \oplus t(\Box p \to p) \subseteq \mathbf{S4} \otimes (\mathbf{K} \oplus \Box_M p \to p) \subseteq (\mathbf{S4} \otimes \mathbf{K}) \oplus t(\Box p \to p) \oplus Mix$$

and the last equation follows from it.

Let us recall now two facts from [22] and [13].

Fact 1. If an intermediate logic $\mathbf{Int} + \Gamma$ has one of the properties listed in the formulation of Theorem 13 then its smallest modal companion $\mathbf{S4} \oplus t(\Gamma)$ has this property as well.

Fact 2. If L_1, L_2 are monomodal logics having one of those properties then the fusion $L_1 \otimes L_2$ also enjoys the same property.

With the help of these results we can derive our Theorem from Theorem 11 and that all the systems involved in the left parts of the equations above have, as is well known, the desirable properties. $\dashv$

[1] Here "+" means the closure under modus ponens and substitution.

In the proof above we used the fact that the BM-companions of the IM-logics under consideration were fusions of monomodal logics with the properties we needed. However, it is impossible to represent in such a way BM-companions of such systems as, e.g. $\mathbf{IntK4} = \mathbf{IntK} \oplus \Box p \to \Box\Box p$. Now we are going to develop a method which in a sense is an extension of Fine's technique for proving the finite model property of subframe logics above **K4**.

From now on we assume all accessibility relations in frames to be *transitive*.

Before describing our construction for BM-frames we briefly remind the reader of the maximal point method used in [8] for the monomodal case. Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ be a model based on a descriptive frame $\mathfrak{F} = \langle W, R, P \rangle$ and φ a formula. Denote by $\mathbf{Sub}\varphi$ the set of φ's subformulas. We say that points $x, y \in W$ are *φ-equivalent* in $\mathfrak{M}$ ($x \sim_\varphi y$ in symbols) if $x \models \psi$ iff $y \models \psi$, for every $\psi \in \mathbf{Sub}\varphi$. Denote by $[x]$ the $\sim_\varphi$-equivalence class generated by x. Call x an *R-maximal point* in $\mathfrak{M}$ (*relative to* φ) if, for any proper R-successor y of x in $\mathfrak{F}$, $x \not\sim_\varphi y$. A point x is said to be *R-eliminable* in $\mathfrak{F}$ if it has a proper R-successor in every set $X \in P$ containing x. Every R-maximal point in $\mathfrak{M}$ relative to φ is clearly R-noneliminable in $\mathfrak{F}$.

Lemma 14 *Suppose φ is a modal formula and $\mathfrak{M}$ a model based on a descriptive frame $\mathfrak{F} = \langle W, R, P \rangle$. Then for every point x in $\mathfrak{M}$, there is an R-maximal point y (relative to φ) such that $x \sim_\varphi y$ and $x = y$ or xRy. The set of final R-clusters in $\mathfrak{F}$ is non-empty and forms a cover for $\mathfrak{F}$.*

Proof. Follows from [7]. $\dashv$

For a modal frame $\mathfrak{F} = \langle W, R, P \rangle$ and a non-empty set $V \in P$, the structure

$$\langle V, R{\restriction}V, \{X \cap V : X \in P\} \rangle$$

is also a modal frame; it is called the *subframe* of $\mathfrak{F}$ *induced* by V. If V is cofinal in the subframe $\mathfrak{F}'$ of $\mathfrak{F}$ generated by V (i.e., each point in $\mathfrak{F}'$ either "sees" a point in V or belongs to V itself), then the subframe of $\mathfrak{F}$ induced by V is called *cofinal*.

A classical modal logic is said to be a (*cofinal*) *subframe logic* if it is characterized by a class of frames closed under the formation of (cofinal) subframes. **K4**, **K4.3**, **S4**, **GL**, **Grz** and many other standard systems are examples of subframe logics; **K4.1** and **K4.2** are cofinal subframe logics but not subframe ones. All cofinal subframe logics enjoy FMP. Another remarkable feature is that such important properties as d-persistence, compactness, elementarity turn out to be equivalent for cofinal subframe logics. For more information about subframe and cofinal subframe logics consult [8], [20] (for the intransitive case) and [25].

Lemma 15 *Suppose $\mathfrak{F} = \langle W, R, P\rangle$ is a descriptive frame for a logic L and V a set of R-noneliminable points in $\mathfrak{F}$ such that the Kripke frame $\mathfrak{G} = \langle V, R{\restriction}V\rangle$ is of finite depth*[2]*.*

(i) *If L is a subframe logic then $\mathfrak{G} \models L$.*

(ii) *If L is a cofinal subframe logic and $\mathfrak{G}$ a cofinal subframe of $\kappa\mathfrak{F}$ with a finite cover then $\mathfrak{G} \models L$.*

Proof. Follows from [8] and [25]. $\dashv$

Now we extend the method of maximal points to transitive BM-frames.

Let $\mathfrak{F} = \langle W, R_I, R_M, P\rangle$ be a descriptive BM-frame validating Mix and φ a $\mathcal{L}\Box_I\Box_M$-formula which is refuted in $\mathfrak{F}$ under a valuation $\mathfrak{V}$ of φ's variables. Take any R_I-maximal point x (relative to φ) at which φ is false and put $X_0 = \{x\}$. Then we define by induction a sequence of sets X_n, for $n \geq 0$, as follows. Suppose X_n has been already defined.

Case 1: n is even. With each point $y \in X_n$ we associate a minimal set Y_y of R_I-maximal R_I-successors of y such that, for every $z \in W$, if yR_Iz and there is no R_I-successor of y in $X_n \cap [z]$ then Y_y contains a $z' \in [z]$. Put

$$X_{n+1} = X_n \cup \bigcup_{y\in X_n} Y_y. \tag{4}$$

Case 2: n is odd. Now with each $y \in X_n$ we associate a minimal set Y_y of simultaneously R_I- and R_M-maximal R_M-successors of y such that, for every $z \in W$, if yR_Mz and there is no R_M-successor of y in $X_n \cap [z]$ then Y_y contains a $z' \in [z]$. Then again we define X_{n+1} by (4).

The existence of Y_y in Case 1 follows from Lemma 14. As for Case 2, suppose $y \in X_n$ has an R_M-successor $z \in W$ but no R_M-successor $z' \in [z]$ in X_n. Using Lemma 14, choose an R_M-maximal (relative to φ) R_M-successor $z_1 \in [z]$ of y. If it is also R_I-maximal then we may put it in Y_y. Otherwise take an R_I-maximal R_I-successor $z_2 \in [z]$ of z_1. By (3), we have yR_Mz_2. So it remains to show that z_2 is R_M-maximal. Suppose this is not the case. Then take an R_M-maximal R_M-successor $z_3 \in [z]$ of z_2. Clearly $z_1R_Mz_3$ and so both z_1 and z_2 belong to the same R_M-cluster (for otherwise we arrive at a contradiction with z_1 being R_M-maximal and $z_1 \sim_\varphi z_3$). Hence z_1 is R_M-reflexive and then $z_1R_Mz_2$, which is a contradiction.

It should be clear that X_n is finite, for every $n < \omega$. Put

$$X_\omega = \bigcup_{i<\omega} X_i$$

and let $\mathfrak{M}' = \langle\mathfrak{F}', \mathfrak{V}'\rangle$ where $\mathfrak{F}' = \langle X_\omega, R_I{\restriction}X_\omega, R_M{\restriction}X_\omega\rangle$ and $\mathfrak{V}' = \mathfrak{V}{\restriction}X_\omega$.

[2] A transitive frame $\mathfrak{F}$ is said to be of *finite depth* if, for some natural n, every strictly ascending chain in $\mathfrak{F}$ contains $\leq n$ points.

Lemma 16 (i) *X_ω is finite.*

(ii) *For every $\psi \in \mathbf{Sub}\varphi$ and every $y \in X_\omega$,*

$$(\mathfrak{M}, y) \models \psi \text{ iff } (\mathfrak{M}', y) \models \psi.$$

(iii) *All points in X_ω are R_I-maximal relative to φ.*

Proof. (i) Suppose X_ω is infinite. By König's Lemma, we then have a sequence of points $(x_n)_{n<\omega}$ such that, for every natural n, the points x_{2n} are R_M-maximal and $x_{2n} R_I x_{2n+1} R_M x_{2n+2}$. Then using (3), we can extract from it an infinite ascending R_M-chain of R_M-maximal points (relative to φ) belonging to distinct R_M-clusters, which is a contradiction.

(ii) is proved by induction on the construction of ψ and (iii) follows from the construction of X_ω. ⊣

In the obvious way the notion of subframe logic can be extended to BM-logics. We obtain for them the following

Theorem 17 *Suppose $L \supseteq (\mathbf{S4} \otimes \mathbf{K4}) \oplus Mix$ is a d-persistent subframe BM-logic.*

(i) *If $\mathbf{S4} \oplus \Gamma$ is a subframe logic (in the language $\mathcal{L}\Box_I$) then $L \oplus \Gamma$ has FMP.*

(ii) *If $\mathbf{S4} \oplus \Gamma$ is a cofinal subframe logic (in the language $\mathcal{L}\Box_I$) of finite width*[3] *then $L \oplus \Gamma$ has FMP.*

Proof. (i) Suppose $\varphi \notin L \oplus \Gamma$. Take a descriptive frame $\mathfrak{F}$ for $L \oplus \Gamma$ which under some valuation refutes φ at an R_I-maximal point x. Consider the Kripke subframe $\mathfrak{G}$ of $\kappa\mathfrak{F}$ induced by X_ω. By Lemma 16, $\mathfrak{G}$ refutes φ. It remains to show that $\mathfrak{G} \models L \oplus \Gamma$. Since L is a d-persistent subframe logic, $\mathfrak{G} \models L$. And since $\mathbf{S4} \oplus \Gamma$ is a subframe logic, by Lemma 15 we have also $\mathfrak{G} \models \Gamma$.

(ii) At each step in the definition of X_{n+1} together with every new point we add to X_n also a point from each R_I-final R_I-cluster that is R_I-accessible from it. According to [7], since $\mathbf{S4} \oplus \Gamma$ is of finite width, there are finitely many R_I-final R_I-clusters in $\mathfrak{F}$ and they form an R_I-cover for $\mathfrak{F}$. It follows also that X_ω is R_I-cofinal in $\mathfrak{F}$ and finite. As a result the selected subframe will be an R_I-cofinal subframe of $\mathfrak{F}$'s underlying Kripke frame and so we may use Lemma 15 once again. ⊣

It is worth noting that the basic logic $(\mathbf{S4} \otimes \mathbf{K4}) \oplus Mix$ is d-persistent (because it is axiomatized by Sahlqvist formulas, see [17]) and its class of Kripke frames is closed under the formation of subframes. Hence $(\mathbf{S4} \otimes \mathbf{K4}) \oplus Mix$ has FMP.

Corollary 18 *Suppose an IM-logic $L \supseteq$ **IntK4** has a d-persistent subframe BM-companion $M \supseteq (\mathbf{S4} \otimes \mathbf{K4}) \oplus Mix$. Then*

[3]This means that, for some $n < \omega$, it is characterized by a class of frames in which every point has at most n mutually inaccessible successors.

(i) *for every set* Γ *of intuitionistic negation and disjunction free formulas,* $L \oplus \Gamma$ *has FMP;*

(ii) *for every set* Γ *of intuitionistic disjunction free formulas and every* $n \geq 1$, $L \oplus \Gamma \oplus wd_n$ *has FMP, where*

$$wd_n = \bigvee_{i=0}^{n} (p_i \to \bigvee_{j \neq i} p_j).$$

Proof. It is shown in [23] that if an intuitionistic formula φ contains no $\neg$ and $\vee$ then $\mathbf{S4} \oplus t(\varphi)$ is a subframe logic and if φ has no $\vee$ then $\mathbf{S4} \oplus t(\varphi)$ is a cofinal subframe one. It is well known also that $\mathbf{S4} \oplus t(wd_n)$ is the minimal logic above $\mathbf{S4}$ of width n. The claim of the Corollary follows now immediately from Theorems 17 and 11. $\dashv$

Example 19 As a consequence of Corollary 18 we obtain that the following IM-logics have FMP:

(1) $\mathbf{IntK4}$;

(2) $\mathbf{IntS4} = \mathbf{IntK4} \oplus \Box p \to p$ (R is reflexive);

(3) $\mathbf{IntS4.3} = \mathbf{IntS4} \oplus \Box(\Box p \to q) \vee \Box(\Box q \to p)$ (R is reflexive and connected);

(4) $\mathbf{IntK4} \oplus p \vee \Box\neg\Box p$ (R is symmetrical);

(5) $\mathbf{IntK4} \oplus \Box p \vee \Box\neg\Box p$ (R is Euclidean);

(6) $\mathbf{IntK4} \oplus \Box p \vee \neg\Box p$ ($x \leq y \wedge xRz \Rightarrow yRz$);

(7) $\mathbf{IntK4} \oplus p \to \Box p$ ($xRy \Rightarrow x \leq y$);

(8) $\mathbf{IntK4} \oplus \Box(p \vee \neg p)$ ($xRy \wedge y \leq z \Rightarrow z = y$);

(9) $\mathbf{IntK4} \oplus \Box p \vee \Box\neg p$ ($xRy \wedge xRz \Rightarrow y = z$);

(10) $\mathbf{IntK4} \oplus \Box(p \to q) \vee \Box(q \to p)$ ($xRy \wedge xRz \Rightarrow y \leq z \vee z \leq y$).

Proof. It is shown in [19] that the logics listed above are characterized by the classes of Kripke frames satisfying the corresponding first order conditions in brackets. For each logic L in the list, one can write down a (bimodal) Sahlqvist formula φ having a universal first order equivalent and such that

$$\rho((\mathbf{S4} \otimes \mathbf{K4}) \oplus Mix \oplus \varphi) = L.$$

For instance,

$$\rho((\mathbf{S4} \otimes \mathbf{S4.3}) \oplus Mix) = \mathbf{IntS4.3},$$

$$\rho((\mathbf{S4} \otimes \mathbf{K4}) \oplus Mix \oplus p \to \Box_M \Diamond_M p) = \mathbf{IntK4} \oplus p \vee \Box\neg\Box p,$$

$$\rho((\mathbf{S4} \otimes \mathbf{K4}) \oplus Mix \oplus \Diamond_M \Box_M p \to \Box_M p) = \mathbf{IntK4} \oplus \Box p \vee \Box\neg\Box p,$$

$$\rho((\mathbf{S4} \otimes \mathbf{K4}) \oplus Mix \oplus \Diamond_I \Box_M p \to \Box_M p) = \mathbf{IntK4} \oplus \Box p \vee \neg\Box p.$$

Since logics axiomatized by Sahlqvist formulas are d-persistent and since logics characterized by universal classes of Kripke frames are subframe ones, the constructed BM-companions of the listed IM-logics satisfy the requirements of Corollary 18 and so the latter have FMP. This property will be preserved if we extend these logics with intuitionistic formulas meeting the conditions of Corollary 18 (i) or (ii). $\dashv$

Corollary 20 *For every variable free $\mathcal{L}\Box$-formula φ, the logic* $\mathbf{IntK4} \oplus \varphi$ *has FMP.*

Proof. Using the Deduction Theorem and reducing modalities by Mix, we clearly have that, for every $\mathcal{L}\Box_I\Box_M$-formula ψ,

$$\psi \in (\mathbf{S4} \otimes \mathbf{K4}) \oplus t(\varphi) \oplus Mix \text{ iff } t(\varphi) \wedge \Box_M t(\varphi) \to \psi \in (\mathbf{S4} \otimes \mathbf{K4}) \oplus Mix.$$

It follows that the logic $(\mathbf{S4}\otimes\mathbf{K4})\oplus t(\varphi)\oplus Mix$ has FMP and so, by Theorem 11, its IM-fragment $\mathbf{IntK4} \oplus \varphi$ has this property as well. $\dashv$

Theorem 21 *Suppose $L \supseteq (\mathbf{S4} \otimes \mathbf{K4}) \oplus Mix$ is a d-persistent BM-logic whose class of Kripke frames is closed under forming R_M-cofinal subframes.*

(i) *If* $\mathbf{S4} \oplus \Gamma$ *is a subframe logic then $L \oplus \Gamma$ has FMP;*

(ii) *If* $\mathbf{S4} \oplus \Gamma$ *is a cofinal subframe logic of finite width then $L \oplus \Gamma$ has FMP.*

Proof. (i) Let $\varphi \notin L \oplus \Gamma$. Take any descriptive frame $\mathfrak{F}$ for $L \oplus \Gamma$ refuting φ under a valuation $\mathfrak{V}$ of φ's variables and pick in $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ an R_I-maximal point x (relative to φ) at which φ is false. Since $\mathfrak{F}$ is descriptive, its every point "sees" via R_M an R_M-final R_M-cluster. Let $\{C_j : j \in J\}$ be the set of all such clusters. By Mix, every C_j is closed under R_I, i.e., if $x \in C_j$ and xR_Iy then $y \in C_j$.

For every $j \in J$, fix an R_I-maximal (relative to φ) point $x_j \in C_j$ and put

$$X_0 = \{x\} \cup \{x_j : j \in J\}.$$

Now, starting from X_ω in exactly the same way as before we form the sets X_n, for $n < \omega$, and X_ω. But unlike the previous construction, the new X_ω is not necessarily finite. It is essential, however, that

(iv) the frame $\mathfrak{F}' = \langle X_\omega, R_I\restriction X_\omega, R_M\restriction X_\omega \rangle$ is of finite depth and

(v) there is a constant $c > 0$ such that $|X_\omega \cap C_j| \leq c$, for every $j \in J$.

Let $\mathfrak{V}' = \mathfrak{V}\restriction X_\omega$ and $\mathfrak{M}' = \langle \mathfrak{F}', \mathfrak{U}' \rangle$. By Lemma 16 (ii), which clearly still holds, $\mathfrak{M}' \not\models \varphi$. Besides, since L is d-persistent, $\mathfrak{F}' \models L$ and, by (iv) and Lemma 15, $\mathfrak{F}' \models \Gamma$. However, this proves only that $L \oplus \Gamma$ is Kripke complete. To establish FMP, we reduce $\mathfrak{M}'$ to a finite model separating φ from $L \oplus \Gamma$.

Define an equivalence relation $\sim$ on the set $\{Y_j = C_j \cap X_\omega : j \in J\}$ by taking $Y_i \sim Y_j$ iff the submodels of $\mathfrak{M}'$ induced by Y_i and Y_j are isomorphic. By (v), there are only finitely many pairwise $\sim$-nonequivalent $Y_i's$. Now, identifying $\sim$-equivalent sets in $\mathfrak{M}'$ and defining the accessibility relations for $\Box_I$ and $\Box_M$ in the canonical way, we clearly obtain a finite model which is a p-morphic image of $\mathfrak{M}'$.

(ii) is proved in the same manner as (ii) in Theorem 17. ⊣

Corollary 22 *Suppose an IM-logic* $L \supseteq$ **IntK4** *has a d-persistent BM-companion* $M \supseteq (\mathbf{S4} \otimes \mathbf{K4}) \oplus Mix$ *whose class of Kripke frames is closed under the formation of* R_M*-cofinal subframes. Then the claims (i) and (ii) of Corollary 18 hold.*

Example 23 Using Corollary 22 in the same way as in Example 19 we can establish FMP of $\mathbf{IntS4} \oplus \neg\Box p \vee \neg\Box\neg p$, $\mathbf{IntS4} \oplus \Box\neg\Box p \vee \Box\neg\Box\neg p$, and many other logics.

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