The product of converse PDL and polymodal K

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Abstract

The product of two modal logics L_1 and L_2 is the modal logic determined by the class of frames of the form $\mathcal{F} \times \mathcal{G}$ such that $\mathcal{F}$ and $\mathcal{G}$ validate L_1 and L_2, respectively. This paper proves the decidability of the product of converse **PDL** and polymodal **K**. Decidability results for products of modal logics of knowledge as well as temporal logics and polymodal **K** are discussed. All those products form rather expressive but still decidable fragments of modal predicate logics. Based on the equivalence of polymodal **K** and the description logic $\mathcal{ALC}$ we shall discuss the obtained fragments, extend the expressive power a bit, and compare them with other modal description logics.

1 Introduction

This paper investigates the decision problem for propositional modal logics interpreted in products of Kripke frames: the product

$$\mathcal{F} \times \mathcal{G} = \langle W \times V, \underline{R_1}, \ldots, \underline{R_n}, \underline{S_1}, \ldots, \underline{S_m} \rangle$$

of two Kripke frames $\mathcal{F} = \langle W, R_1, \ldots, R_n \rangle$ and $\mathcal{G} = \langle V, S_1, \ldots, S_m \rangle$ is defined by putting

$$\langle w, v \rangle \underline{R_i} \langle w', v' \rangle$$

iff $v = v'$ and wR_iw' and

$$\langle w, v \rangle \underline{S_i} \langle w', v' \rangle$$

iff $w = w'$ and vS_iv'. The product $L_1 \times L_2$ of two modal logics L_1 and L_2 is the logic determined by the class of products $\mathcal{F} \times \mathcal{G}$ such that $\mathcal{F}$ and $\mathcal{G}$ validate L_1 and L_2, respectively. In contrast to "flat" modal logics products enable us to model different dimensions of an application domain in one formalism, a typical example is a domain comprising both a spatial and a temporal dimension. We mention also the "temporal logics of agency" in Fagin et al. (1995), where a temporal dimension is combined with a dimension for the states which are believed to be possible by certain agents, and (fragments of) modal predicate logics, where the "object–dimension" is represented by models of first order predicate logic and the intensional (or dynamic, or temporal) dimension by means of accessibility relations interpreting modal operators. We refer the reader to Gabbay and Shehtman (1998) for more examples.

Consequently, products of modal logics have been investigated intensively in the last years, see e.g. Marx and Venema (1997), Gabbay and Shehtman (1998), Reynolds (1996), (1997), and Marx (1997). It turned out that from a technical viewpoint products of modal logics are considerably more complex than their "flat" companions. We are alluding here to various undecidability results for products of rather well behaved modal logics, e.g., the products $\mathbf{K4.3} \times \mathbf{K4.3}$ (the bimodal logic determined by frames of the form $\mathcal{F} \times \mathcal{G}$ such that $\mathcal{F}$ and $\mathcal{G}$ are transitive and weakly linear) and $\mathbf{K}_u \times \mathbf{K}_u$ (the 4–modal logic determined by products of structures of the form $\langle W, R, W \times W \rangle$) are undecidable, see Marx (1997) and Reynolds (1997). Positive results turned out to be hard to obtain, we refer the reader to the decidability proof for the products $\mathbf{K}_m \times \mathbf{K}_m$ and $\mathbf{S5} \times \mathbf{K}_m$ in Gabbay and Shehtman (1998)[1]; here $\mathbf{K}_m$ denotes polymodal $\mathbf{K}$ with m modal operators.

In this paper we are going to prove the decidability of products of the form $L \times \mathbf{K}_m$, for a number of expressive modal logics L. The main result states that the product of converse **PDL** and $\mathbf{K}_m$ is decidable. Recall that propositional dynamic logic **PDL** and its extensions like converse **PDL** were indroduced for reasoning about the behavior of non–determistic programs, see Harel (1984). This formalism turned out to be useful also as a basis for a logic of action and for deontic logic, see Segerberg (1980), Prendinger and Schurz (1996), de Giacomo and Lenzerini (1995), and Meyer (1988).

The decidability of various other logics interpreted in products of Kripke frames follows immediately from our result, or can be proved by means of straightforward modifications of the method presented: the decidability of the products $\mathbf{S4} \times \mathbf{K}_m$ and $\mathbf{K4} \times \mathbf{K}_m$ is an obvious corollary. The product $\mathbf{S5}_n^C \times \mathbf{K}_m$ of the modal logic of knowledge with Common-knowledge operators $\mathbf{S5}_n^C$ and $\mathbf{K}_m$ is embedded in our system, and so decidable. Moreover, for a number of products of temporal logics with polymodal $\mathbf{K}$ we obtain decidability by combining the method presented in the present paper with the technique developed in Wolter and Zakharyaschev (1998c). We mention that, in contrast to the methodology presented in Gabbay and Shehtman (1998), here we do not prove the finite model property in order to establish decidability. Most of the products of temporal logics and polymodal $\mathbf{K}$ actually do not have the finite model

[1]We refer the reader to Gabbay and Shehtman (1998) also for a comprehensive survey of positive and negative results on the decidability of products of modal logics.

property and it is open whether the product of converse **PDL** and $\mathbf{K}_m$ is determined by its finite models. The decidability proof developed in the present paper is based on two ideas: the first ingredient is the notion of a quasimodel for a formula φ. It enables us to encode the $\mathbf{K}_m$–dimension of a model based on a product by means of finite labelled trees in such a way that the resulting quasimodels are equivalent—modulo φ—to the original ones. Quasimodels for a given formula φ are not finite, and it is easy to see that the logics considered in the present paper do not have the finite model property with respect to quasimodels. So a second step is required to provide a decision procedure. It consists in the proof that quasimodels can be decomposed into a finite set of finite blocks in such a way that it can be recognized effectively whether a given set of blocks results from a decomposition of a quasimodel or not. So the second step is closely related to the mosaic–method (see Nemeti (1995)) and to the methods developed in Wolter (1997), (1998) to prove the decidability of temporal logics and bimodal provability logics without the finite model property. The quasimodels introduced here form a variant of those developed in Wolter and Zakharyaschev (1998b).

The listed products can be interpreted in different ways. For example, $\mathbf{S5}_n^C \times \mathbf{K}$ is a fragment of the temporal logic of agency of Fagin et al. (1995) in which agents do not forget, do not learn, and know time. In contrast to the full system it is decidable. Our main interest in products $L \times \mathbf{K}_m$, however, relies on the fact that they form natural fragments of modal predicate logics: recall that $\mathbf{K}_m$ can be regarded as a fragment of first order predicate logic, so products of the form $L \times \mathbf{K}_m$, L a modal logic, are fragments of modal predicate logics the modal part of which coincides with L. More precisely, they are fragments of modal predicate logics based on constant domains with globally interpreted binary predicates (the accessibility relations). In the final section of this paper it is shown that this condition can be dispensed with; that is to say, we show that decidability is preserved for our fragments when both global as well as local interpretations of binary predicates are allowed. In this way we obtain rather expressive but still decidable fragments of modal predicate logics: the product of converse **PDL** and polymodal **K** forms a decidable basis for a logic of action, $\mathbf{S5}_n^C \times \mathbf{K}_m$ is a rather expressive logic of knowledge, and products of temporal logics with $\mathbf{K}_m$ are decidable formalisms for representing processes.

Recently similar fragments of modal predicate logics have received considerable interest in description logic. Description logics are knowledge representation languages in the style of KL-ONE which can be used to define the concepts (alias unary predicates) of an application domain as well as the interaction between the concepts, see Donini et al. (1996), Brachman and Schmolze (1985). To this end, in the standard description language $\mathcal{ALC}$, complex concepts are constructed from primitive ones and roles (alias binary predicates) with the help of boolean operators and restricted quantification (see Donini et al. (1991) and Schmidt–Schauß and Smolka (1991)). $\mathcal{ALC}$ and most of its variants are suitable only to represent static, time–independent facts. In order to add to this language a temporal (or dynamic, or epistemic) dimension various extensions of $\mathcal{ALC}$ by means of modal operators have been proposed (see e.g. Schmiedel (1990), Schild (1993), Gräber et al. (1995), Laux (1994), Baader and Laux (1995), Baader and Ohlbach (1995), Wolter and Zakharyaschev (1998a), (1998b), (1998c)). $\mathcal{ALC}$ is just a syntactic variant of polymodal **K** and so the modal logics we investigate are syntactic variants of description logics extended by means of modal operators. To make a comparison of the logics introduced in the present paper and modal description logics from the literature more transparent, the last section of this paper is written in the terminology of description logics.

The paper is organized in the following manner: the second section introduces the syntax and semantics of the product of converse **PDL** and polymodal **K**. Sections 3, 4, and 5 together establish the decidability of this product. In section 4 the notion of a quasimodel is introduced and section 5 establishes the decidability of the product by applying a variant of the mosaic–technique to quasimodels. In section 6 the decidability of related systems is discussed. In section 7 the notation of description logic is introduced, an extension of the decidability result of section 5 is proved, and the modal description logics introduced in the present paper are compared with other options. The final section formulates various further research directions.

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2 Syntax and Semantics

In this section we are going to introduce the syntax and semantics of the logic $\mathbf{PDLK}_m$.

DEFINITION 2.1 (Alphabet) The language $\mathcal{L}_{\mathbf{PDLK}_m}$ has the following alphabet:

- a set of propositional variables: $p_0, p_1, \ldots$,
- the booleans: $\wedge$ and $\neg$,
- modal operators: $\Box_1, \ldots, \Box_m$,
- action variables: $\alpha_1, \alpha_2 \ldots$,
- action term constructors: $\cup$ (alternation), ; (composition), * (iteration), ? (test), $^-$ (converse).

DEFINITION 2.2 The formulas and action terms of the language $\mathcal{L}_{\mathbf{PDLK}_m}$ are defined by simultanous induction:

- all propositional variables are formulas,
- if φ and ψ are formulas and α is an action term, then $\neg\varphi$, $\Box_i\varphi$, $[\alpha]\varphi$, and $\varphi \wedge \psi$ are formulas,
- all action variables α_i as well as the converses α_i^- are action terms,
- if α and β are action terms, then $\alpha;\beta$, $\alpha \cup \beta$, α^* are action terms,
- if φ is a formula, then $\varphi?$ is an action term.

Observe that we allow the application of the converse operator to action variables only. The application of the converse operator to arbitrary action terms is introduced as an abbreviation. Define inductively: $(\alpha_1;\alpha_2)^- = \alpha_2^-;\alpha_1^-$, $(\alpha_1 \cup \alpha_2)^- = \alpha_1^- \cup \alpha_2^-$, $(\alpha^*)^- = (\alpha^-)^*$, $(\psi?)^- = \psi?$.

Formulas of $\mathcal{L}_{\mathbf{PDLK}_m}$ are interpreted in products

$$\mathcal{F} \times \mathcal{G} = \langle W \times \Delta, \underline{T_{\alpha_1}}, \ldots, \underline{R_1}, \ldots, \underline{R_m} \rangle$$

of **PDL**–structures

$$\mathcal{F} = \langle W, T_{\alpha_1}, T_{\alpha_2}, \ldots \rangle,$$

where the T_{α_i} are binary relations for any action variable α_i, and $\mathbf{K}_m$-structures

$$\mathcal{G} = \langle \Delta, R_1, \ldots, R_m \rangle.$$

DEFINITION 2.3 A **PDLK**-valuation $\mathcal{V}$ into $\mathcal{F} \times \mathcal{G}$ is a mapping from the set of propositional variables into $2^{W \times \Delta}$. By simultanous induction we define the satisfaction–relation $\models$ between pairs $\langle w, x \rangle \in W \times \Delta$ and formulas and the relations $\underline{T_\alpha} \subseteq (W \times \Delta) \times (W \times \Delta)$, α an action term:

- $\langle w, x \rangle \models p$ iff $\langle w, x \rangle \in \mathcal{V}(p)$.
- $\langle w, x \rangle \models \varphi \wedge \psi$ iff $\langle w, x \rangle \models \varphi$ and $\langle w, x \rangle \models \psi$.
- $\langle w, x \rangle \models \neg\varphi$ iff $\langle w, x \rangle \not\models \varphi$.
- $\langle w, x \rangle \models \Box_i\varphi$ iff $\langle w, y \rangle \models \varphi$, for all $y \in \Delta$ with xR_iy.
- $\langle w, x \rangle \models [\alpha]\varphi$ iff $\langle v, x \rangle \models \varphi$, for all $v \in W$ with $\langle w, x \rangle \underline{T_\alpha} \langle v, x \rangle$.
- $\underline{T_{\alpha;\beta}} = \underline{T_\alpha} \circ \underline{T_\beta}$.
- $\underline{T_{\alpha\cup\beta}} = \underline{T_\alpha} \cup \underline{T_\beta}$.
- $\underline{T_{\alpha^*}} = \underline{T_\alpha}^*$ (the reflexive and transitive closure).
- $\underline{T_{\alpha_i^-}} = \underline{T_{\alpha_i}}^{-1}$ (the converse relation).
- $\underline{T_{\psi?}} = \{\langle\langle w, x \rangle, \langle w, x \rangle\rangle : \langle w, x \rangle \models \psi\}$.

A formula φ is satisfiable iff there exists a product $\mathcal{F} \times \mathcal{G}$ with a valuation $\mathcal{V}$ and $\langle w, x\rangle \in W \times \Delta$ such that $\langle w, x\rangle \models \varphi$. The set of formulas φ such that $\neg\varphi$ is not satisfiable is denoted by $\mathbf{PDLK}_m$.

Observe that the relation $\underline{T_\alpha}$ does not depend on the Δ–dimension whenever α does not contain "test". We are in the position now to formulate the main result of this paper:

THEOREM 2.4 $\mathbf{PDLK}_m$ *is decidable.*

Observe that this result does not extend to the global consequence relation $\models^*$ associated with $\mathbf{PDLK}_m$. Here, for formulas φ and ψ we write $\varphi \models^* \psi$ iff, for all models $\mathcal{F} \times \mathcal{G}$,

$$\forall\langle w, x\rangle \in W \times \Delta \ \langle w, x\rangle \models \varphi \quad \Rightarrow \quad \forall\langle w, x\rangle \in W \times \Delta \ \langle w, x\rangle \models \psi.$$

The global consequence relation associated with $\mathbf{K} \times \mathbf{K}$ is undecidable, see Marx (1997), hence the global consequence associated with $\mathbf{PDLK}_m$ is undecidable as well.

In what follows we shall assume for simplicity that we have only one modal operator $\Box$ in the $\mathbf{K}$–dimension. The language $\mathcal{L}_{\mathbf{PDLK}_1}$ and the logic $\mathbf{PDLK}_1$ are denoted by $\mathcal{L}_{\mathbf{PDLK}}$ and $\mathbf{PDLK}$, respectively. This simplification will pay off in the rather technical proof of Theorem 2.4, but it will be clear how to extend the results to $\mathbf{K}_m$.

We close this section with the observation that $\mathbf{PDLK}$ does not have the *product finite model property*; that is to say, there exists a formula φ which is satisfiable in an infinite product but not in a finite one.

THEOREM 2.5 $\mathbf{PDLK}$ *does not have the product finite model property.*

Proof Let φ be the conjunction of

$$[\alpha_1^*]\Diamond p, \ \ [\alpha_1^*]\Box(p \to \langle\alpha_1^*\rangle[\alpha_1^*]\neg p).$$

(Here and in what follows $\Diamond\psi$ and $\langle\alpha\rangle\psi$ abbreviate $\neg\Box\neg\psi$ and $\neg[\alpha]\neg\psi$, respectively.) Let $\mathcal{F} = \langle \mathbf{N} - \{0\}, T_{\alpha_1}\rangle$, where $\mathbf{N}$ denotes the set of natural numbers and

$$iT_{\alpha_1}j \Leftrightarrow j = i + 1.$$

Let $\mathcal{G} = \langle \mathbf{N}, R\rangle$, where xRy iff $x = 0$ and $y \neq 0$. Let $\mathcal{V}$ be a valuation in $\mathcal{F} \times \mathcal{G}$ defined by $\mathcal{V}(p) = \{\langle i, i\rangle : i \in \mathbf{N} - \{0\}\}$. Then $\langle 1, 0\rangle \models \varphi$ but it is easy to see that φ is not satisfiable in a finite product. ❑

Observe that φ contains the operator $[\alpha_1^*]$ only. Hence also $\mathbf{S4} \times \mathbf{K}$ does not have the product finite model property.

3 Trees

In this section we prepare the proof of Theorem 2.4. The depth $dp(x)$ of a point x in an intransitive tree $\mathcal{K} = \langle \Delta, R \rangle$ is the length of the path from the root of $\langle \Delta, R \rangle$ to x. If the set $\{dp(x) : x \in \Delta\}$ is bounded, then the depth $dp(\mathcal{K})$ of $\mathcal{K}$ is defined by putting

$$dp(\mathcal{K}) = \max\{dp(x) : x \in \Delta\}.$$

DEFINITION 3.1 The depth $d(\varphi)$ of a formula φ and $d(\alpha)$ of a action term α is defined inductively as follows:

$$\begin{array}{rcl} d(p) & = & 0 \\ d(\varphi \wedge \psi) & = & \max\{d(\varphi), d(\psi)\} \\ d(\neg\varphi) & = & d(\varphi) \\ d(\Box\varphi) & = & d(\varphi) + 1 \\ d([\alpha]\varphi) & = & d(\alpha) + d(\varphi) \\ d(\alpha_i) = d(\alpha_i^-) & = & 0 \\ d(\alpha;\beta) & = & \max\{d(\alpha), d(\beta)\} \\ d(\alpha \cup \beta) & = & \max\{d(\alpha), d(\beta)\} \\ d(\alpha^*) & = & d(\alpha) \\ d(\psi?) & = & d(\psi) \end{array}$$

THEOREM 3.2 *Suppose φ is satisfiable. Then φ is satisfiable in a product $\mathcal{F} \times \mathcal{K}$ such that $\mathcal{F}$ is a* **PDL***-structure and $\mathcal{K}$ is an intransitive tree of depth $\leq d(\varphi)$.*

Proof The proof applies the unravelling technique (we refer the reader to Chagrov and Zakharyaschev (1997) and Gabbay and Shehtman (1998), where this technique is also applied to investigate products.) We give a sketch only: assume that $\mathcal{V}$ is a valuation in $\mathcal{F} \times \mathcal{G}$ such that $\langle w, x_0 \rangle \models \varphi$. Let $\mathcal{G} = \langle \Delta, R \rangle$ and $m = d(\varphi)$. Then we put $\mathcal{K} = \langle \Delta', R' \rangle$, where

$$\Delta' = \{\langle x_0, \ldots, x_k \rangle : k \leq m,\ \forall i < k(x_i R x_{i+1})\},$$

$$\langle x_0, \ldots, x_k \rangle R' z \Leftrightarrow \quad \exists y\ (z = \langle x_0, \ldots, x_k, y \rangle).$$

The structure $\mathcal{K}$ is an intransitive tree of depth $\leq d(\varphi)$. Define a valuation $\mathcal{V}'$ in $\mathcal{F} \times \mathcal{K}$ by putting

$$\langle w, \langle x_0, \ldots, x_k \rangle \rangle \in \mathcal{V}'(p) \Leftrightarrow \langle w, x_k \rangle \in \mathcal{V}(p).$$

It is easy to check that for all ψ with $d(\psi) \leq m$ and all i with $0 \leq i \leq m - d(\psi)$:

$$\langle w, \langle x_0, \ldots, x_i \rangle \rangle \models_{\mathcal{V}'} \psi \Leftrightarrow \langle w, x_i \rangle \models_{\mathcal{V}} \psi.$$

So $\langle w, \langle x_0 \rangle \rangle \models_{\mathcal{V}'} \varphi$. ❑

4 Quasimodels

In this section we introduce the notion of a quasimodel for a given formula φ. As usual for the investigation of logics related to **PDL** we require a modified notion of the set of subformulas of a formula:

DEFINITION 4.1 (Fischer–Ladner–Closure) The Fischer–Ladner–Closure $sub(\varphi)$ of a formula φ is the smallest set of formulas closed under (ordinary) subformulas satisfying:

- $[\alpha;\beta]\psi \in sub(\varphi)$ implies $[\alpha][\beta]\psi \in \varphi$,
- $[\alpha \cup \beta]\psi \in sub(\varphi)$ implies $[\alpha]\psi \in sub(\varphi)$ and $[\beta]\psi \in sub(\varphi)$,
- $[\alpha^*]\psi \in sub(\varphi)$ implies $[\alpha][\alpha^*]\psi \in sub(\varphi)$,
- $[\alpha_i^-]\psi \in sub(\varphi)$ implies $[\alpha_i]\psi \in sub(\varphi)$,
- $[\psi?]\chi \in sub(\varphi)$ implies $\psi \in sub(\varphi)$.

DEFINITION 4.2 (type) Let φ be a formula. A φ–type t is a subset of $sub(\varphi)$ satisfying the following conditions:

- $\neg\psi \in t$ iff $\psi \notin t$, for all $\neg\psi \in sub(\varphi)$,
- $\varphi \wedge \psi \in t$ iff $\varphi \in t$ and $\psi \in t$, for all $\varphi \wedge \psi \in sub(\varphi)$,
- $[\alpha;\beta]\psi \in t$ iff $[\alpha][\beta]\psi \in t$, for all $[\alpha;\beta]\psi \in sub(\varphi)$,
- $[\alpha^*]\psi \in t$ iff $\psi \in t$ and $[\alpha][\alpha^*]\psi \in t$, for all $[\alpha^*]\psi \in sub(\varphi)$,
- $[\alpha \cup \beta]\psi \in t$ iff $[\alpha]\psi \in t$ and $[\beta]\psi \in t$, for all $[\alpha \cup \beta]\psi \in sub(\varphi)$.
- $[\chi?]\psi \in t$ iff $\psi \in t$ or $\chi \notin t$, for all $[\chi?]\psi \in sub(\varphi)$.

The set of φ–types is denoted by $\mathbf{T}(\varphi)$.

DEFINITION 4.3 (quasistate) A quasistate $\mathfrak{q}$ for φ is a structure

$$\langle\langle T, <\rangle, l\rangle$$

such that $\mathcal{T} = \langle T, <\rangle$ is a finite intransitive tree of depth $\leq d(\varphi)$ and l is a labelling function associating with each member x of T an element $l(x)$ of $\mathbf{T}(\varphi)$ such that the following conditions hold:

- For all $x \in T$ and $\Box\psi \in sub(\varphi)$ we have:

$$\Box\psi \in l(x) \text{ iff } \forall y \in T(x < y \Rightarrow \psi \in l(y)).$$

- For all x, x_1, x_2 in T with $x < x_1$, $x < x_2$, and $x_1 \neq x_2$, the subtrees $\langle \mathcal{T}(x_1), l \rangle$ and $\langle \mathcal{T}(x_2), l \rangle$ of $\mathcal{T}$ generated by x_1 and x_2, respectively, are not isomorphic[2].

Let $n(\varphi) = 2^{|sub(\varphi)|}$ be an upper bound for the number of different φ–types. Define inductively:

$$n_0(\varphi) = n(\varphi), \quad n_{i+1}(\varphi) = n(\varphi) \cdot 2^{n_i(\varphi)}.$$

Then $b(\varphi) := n_{d(\varphi)}(\varphi)$ is an upper bound for the number of different quasistates for φ. The number of points in a quasistate is bounded by $p(\varphi) = 2 \cdot b(\varphi)^{d(\varphi)}$. In what follows we assume that the domains of non-isomorphic quasistates are disjoint and that isomorphic quasistates actually coincide.

DEFINITION 4.4 (basic structure for φ) Let φ be a formula. Consider a **PDL**–structure $\mathcal{F} = \langle W, T_{\alpha_1}, \ldots, T_{\alpha_n} \rangle$ (where $\alpha_1, \ldots, \alpha_n$ is an enumeration of all action variables in φ) with a function q associating with each $w \in W$ a quasistate

$$q(w) = \langle \langle T_w, <_w \rangle, l_w \rangle$$

for φ such that the depth of all $q(w)$ equals m. Then we say that $\langle \mathcal{F}, q \rangle$ is a *basic structure* for φ of depth m.

DEFINITION 4.5 (run, suitable sequence) Let $\langle \mathcal{F}, q \rangle$ be a basic structure for φ of depth m and let $k \leq m$.

A k–run r is a function associating with each $w \in W$ an $r(w) \in T_w$ of depth k. Every k–run is called a run.

Let r, r' be runs such that $r(w) <_w r'(w)$ holds for all $w \in W$. Then r' is called a companion of r, in symbols $r < r'$.

Consider a sequence $\mathcal{R} = \langle \mathcal{R}_0, \ldots, \mathcal{R}_m \rangle$ such that all $r \in \mathcal{R}_k$ are k–runs, for $0 \leq k \leq m$. Then $\mathcal{R}$ is called suitable iff

(S1) $\mathcal{R}_0$ consists of the function which associates with each $w \in W$ the root of T_w.

(S2) For all $r' \in \mathcal{R}_{k+1}$ there exists $r \in \mathcal{R}_k$ with $r < r'$.

(S3) For all $r \in \mathcal{R}_k$, $w \in W$, and $x \in T_w$ with $r(w) <_w x$ there exists $r' \in \mathcal{R}_{k+1}$ such that $r'(w) = x$ and $r < r'$.

In what follows we denote the run in $\mathcal{R}_0$ by r_0.

Let **P** be a property of runs. A sequence $\mathcal{R}$ is called suitable for **P** iff it is suitable and all members of $\mathcal{R}$ have property **P**.

[2] In this definition the generated subtree $\langle \mathcal{T}(x), l \rangle$ of $\mathcal{T}$ is the subtree of $\mathcal{T}$ based on $T(x) = \{y \in T : x <^* y\}$ together with the restriction of l to $T(x)$.

We are going to define the property **P** of runs which enables us to define quasimodels. To this end define, for each run r in $\langle \mathcal{F}, q \rangle$, the relations $T_\alpha(r)$ on W as follows:

- $wT_{\alpha_i}(r)v$ iff $wT_{\alpha_i}v$,
- $wT_{\alpha_i^-}(r)v$ iff $vT_{\alpha_i}(r)w$,
- $wT_{\alpha;\beta}(r)v$ iff $w(T_\alpha(r) \circ T_\beta(r))v$,
- $wT_{\alpha \cup \beta}(r)v$ iff $w(T_\alpha(r) \cup T_\beta(r))v$,
- $wT_{\alpha^*}(r)v$ iff $wT_\alpha(r)^*v$,
- $wT_{\psi?}(r)v$ iff $w = v$ and $\psi \in l_w(r(w))$.

Observe that (similar to the definition of $\underline{T_\alpha}$) the relation $T_\alpha(r)$ depends on r only when α contains "test".

DEFINITION 4.6 (completely saturated runs) Call a run r completely saturated iff the following holds for all $v \in W$ and all $[\alpha]\psi \in sub(\varphi)$:

- $[\alpha]\psi \in l_v(r(v))$ iff $\forall w \in W(vT_\alpha(r)w \Rightarrow \psi \in l_w(r(w)))$.

If r satisfies the direction from left to right only, then it is called a weakly saturated run.

DEFINITION 4.7 (quasimodel) A basic structure $\langle \mathcal{F}, q \rangle$ for φ is called a quasimodel iff there exists a suitable sequence $\mathcal{R}$ in it for completely saturated runs.

A formula φ is satisfied in a quasimodel $\langle \mathcal{F}, q \rangle$ for φ iff $\varphi \in l_w(r_0(w))$, for some $w \in W$.

THEOREM 4.8 *For every formula φ, φ is satisfiable iff it is satisfiable in a quasimodel for φ.*

Proof Consider a quasimodel $\langle \mathcal{F}, q \rangle$ for φ of depth m with $\varphi \in l_{w_0}(r_0(w_0))$. Let $\mathcal{F} = \langle W, T_{\alpha_1}, \ldots, T_{\alpha_n} \rangle$, $q(w) = \langle \langle T_w, <_w \rangle, l_w \rangle$, and let $\mathcal{R} = \langle \mathcal{R}_0, \ldots, \mathcal{R}_m \rangle$ be a suitable sequence for completely saturated runs in $\langle \mathcal{F}, q \rangle$. We define $\mathcal{G} = \langle \Delta, R \rangle$ by putting:

$$\Delta = \bigcup \{\mathcal{R}_i : i \leq m\},$$

$$rRr' \Leftrightarrow r < r'.$$

Define a valuation $\mathcal{V}$ in $\mathcal{F} \times \mathcal{G}$ by putting for all propositional variables p, all runs r in Δ and all $w \in W$:

$$\langle w, r \rangle \in \mathcal{V}(p) \Leftrightarrow p \in l_w(r(w)).$$

The following two equivalences are checked by induction for all $w, v \in W$:

- for all $\psi \in sub(\varphi)$ and all $r \in \Delta$:

$$\langle w, r\rangle \models \psi \Leftrightarrow \psi \in l_w(r(w)),$$

- for all action terms α in $sub(\varphi)$ and all $r \in \Delta$:

$$wT_\alpha(r)v \Leftrightarrow \langle w, r\rangle \underline{T_\alpha} \langle v, r\rangle.$$

So $\langle w_0, r_0\rangle \models \varphi$ and φ is satisfiable in $\mathcal{F} \times \mathcal{G}$.

For the converse direction assume that φ is satisfied in

$$\mathcal{F} \times \mathcal{G} = \langle W \times \Delta, \underline{T_{\alpha_1}}, \dots \underline{T_{\alpha_n}}, \underline{R}\rangle,$$

where $\mathcal{F}$ is a **PDL**–structure and $\mathcal{G}$ is a tree of depth $m \leq d(\varphi)$. (The existence of such a model satisfying φ follows from Theorem 3.2.) Assume that $\langle w_0, x_0\rangle \models \varphi$ for the root x_0 of $\mathcal{G}$. We are going to construct a quasimodel $\langle \mathcal{F}, q\rangle$ for φ such that $\varphi \in l_{w_0}(r_0(w_0))$. Put, for $\langle w, x\rangle \in W \times \Delta$

$$t(w, x) = \{\psi \in sub(\varphi) : \langle w, x\rangle \models \psi\}.$$

Clearly $t(w, x)$ is a φ–type. Define, for $w \in W$ equivalence relations $\sim_w^0$ on Δ by putting

$$x \sim_w^0 y \Leftrightarrow t(w, x) = t(w, y).$$

We refine, for $w \in W$, the relation $\sim_w^0$ to an equivalence relation $\sim_w^1$ by induction from m down to 0:

- Put $x \sim_w^1 y$ whenever $dp(x) = dp(y) = m$ and $x \sim_w^0 y$.
- Suppose that $\sim_w^1$ is defined for points of depth $k+1$. Then we put $x \sim_w^1 y$ for $x, y \in \Delta$ with $dp(x) = dp(y) = k$ whenever $x \sim_w^0 y$ and (a) for all $z \in \Delta$ with xRz there exists $z' \in \Delta$ such that $z \sim_w^1 z'$ and yRz', (b) for all $z \in \Delta$ with yRz there exists $z' \in \Delta$ such that $z \sim_w^1 z'$ and xRz'.

Denote, for $w \in W$, by $[x]_w$ the $\sim_w^1$–equivalence class generated by x and put $\Delta_w = \{[x]_w : x \in \Delta\}$. Put $[x]_w R_w [y]_w$ iff there exists $z \in [y]_w$ such that xRz. Observe that the mappings $f_w : x \mapsto [x]_w$ are p–morphisms from $\langle \Delta, R\rangle$ onto $\langle \Delta_w, R_w\rangle$. We now define, by means of unravelling, the trees $\langle T_w, <_w\rangle$:

$$T_w = \{\langle [x_0]_w, \dots, [x_k]_w\rangle : \ k \leq m,\ [x_0]_w R_w [x_1]_w R_w \dots R_w [x_k]_w,\ dp(x_0) = 0\},$$

$$\langle [x_0]_w, \dots, [x_k]_w\rangle <_w z$$

iff there exists x_{k+1} such that $[x_k]_w R_w [x_{k+1}]_w$ and $z = \langle [x_0]_w, \dots, [x_k]_w, [x_{k+1}]_w\rangle$. Let, for each $w \in W$,

$$l_w(\langle [x_0]_w, \dots, [x_k]_w\rangle) = t(w, x_k).$$

Of course, $q(w) = \langle\langle T_w, <_w\rangle, l_w\rangle$ is a quasistate for φ, for any $w \in W$. Moreover, $\varphi \in l_{w_0}(r_0(w_0))$. It remains to show that $\langle\mathcal{F}, q\rangle$ is a quasimodel for φ.

To this end we define a sequence $\mathcal{R} = \langle\mathcal{R}_0, \ldots, \mathcal{R}_k\rangle$ which is suitable for completely saturated runs. For any sequence $\langle x_0, \ldots, x_k\rangle$ in Δ such that $x_0 R \ldots R x_k$ the set $\mathcal{R}_k$ contains the mapping

$$r : w \mapsto \langle [x_0]_w, \ldots, [x_k]_w\rangle,$$

and nothing else. Clearly each $\mathcal{R}_k$ consists only of completely saturated runs. It remains to show that $\mathcal{R}$ is suitable. Items (S1) and (S2) are clear, so we concentrate on (S3). Let $r \in \mathcal{R}_k$, $w \in W$, and $x \in T_w$ with $r(w) <_w x$. We have to show that there exists $r' \in \mathcal{R}_{k+1}$ with $r < r'$ and $r'(w) = x$.

Assume $r : w \mapsto \langle [x_0]_w, \ldots, [x_k]_w\rangle$ and $x = \langle [z_0]_w, \ldots, [z_k]_w, [z_{k+1}]_w\rangle$. There exists x_{k+1} with $x_k R x_{k+1}$ such that $x_{k+1} \in [z_{k+1}]_w$. The run $r' : w \mapsto \langle [x_0]_w, \ldots, [x_k]_w, [x_{k+1}]_w\rangle$ is as required. ❑

5 Decidability

In this section we provide an effective criterion for satisfiability in quasimodels. We apply a variant of the mosaic method (see Nemeti (1995), where this method is developed for fragments of first order logic) to the quasimodels introduced above.

DEFINITION 5.1 (block) Let $\langle\mathcal{F}, q\rangle$ with $\mathcal{F} = \langle W, T_{\alpha_1}, \ldots, T_{\alpha_n}\rangle$ be a finite basic structure for φ such that there exists a $w_0 \in W$ such that

(B1) $T_{\alpha_i} \cap T_{\alpha_j} = \emptyset$, for all $i, j \leq n$ with $i \neq j$,

(B2) $T_{\alpha_i} \cap T_{\alpha_j^-} = \emptyset$, for all $i, j \leq n$,

(B3) $|\{v : \exists\beta \in \{\alpha_i, \alpha_i^- : i \leq n\}\ vT_\beta w\}| \leq 2$, for all $w \in W$ with $w \neq w_0$,

(B4) For any $v \in W$ there exists precisely one sequence $\langle v_0, v_1, \ldots, v_m\rangle$ such that the v_i are mutually different, $w_0 = v_0$, $v = v_m$ and such that for any $i < m$ there exists $\beta \in \{\alpha_j, \alpha_j^- : j \leq n\}$ with $v_i T_\beta v_{i+1}$.

w_0 is called a root of $\mathcal{F}$ and is assumed to be fixed. A run r in $\langle\mathcal{F}, q\rangle$ is root saturated iff

$$\forall[\alpha]\psi \in sub(\varphi)([\alpha]\psi \notin l_{w_0}(r(w_0))) \Rightarrow \exists v \in W(w_0 T_\alpha(r) v \ \&\ \psi \notin l_v(r(v))).$$

Now $\langle\mathcal{F}, q\rangle$ is called a block if

(B5) there exists a sequence $\mathcal{R}$ which is suitable for weakly and root saturated runs in $\langle\mathcal{F}, q\rangle$.

DEFINITION 5.2 (satisfying set) A set $\mathcal{S}$ of blocks for φ is *satisfying* if (i) it contains a block with root w_0 such that $\varphi \in l_{w_0}(r_0(w_0))$ and (ii) for every state w in every block $\langle \mathcal{F}_1, q_1 \rangle$ in $\mathcal{S}$ there exists a block $\langle \mathcal{F}_2, q_2 \rangle$ in $\mathcal{S}$ such that $q_1(w) = q_2(v)$ for the root v of $\mathcal{F}_2$.

Our aim is to show that φ is satisfiable iff there is a satisfying set for φ whose blocks contain at most N states, for some $N < \omega$ effectively determined by φ.

For an action term α without iteration denote by $|\alpha|$ the *length* of α which is defined inductively by taking

- $|\alpha_i| = |\alpha_i^-| = 1$, $|\psi?| = 0$;
- $|\alpha \cup \beta| = \max\{|\alpha|, |\beta|\}$; $|\alpha; \beta| = |\alpha| + |\beta|$.

Now for every $n \geq 0$ and any action term α we put

- $\alpha_i(n) = \alpha_i$, $\alpha_i^-(n) = \alpha_i^-$, $\psi?(n) = \psi?$;
- $(\alpha \cup \beta)(n) = \alpha(n) \cup \beta(n)$;
- $(\alpha; \beta)(n) = \alpha(n); \beta(n)$;
- $\alpha^*(n) = \alpha^{\leq n}(n)$,

where

$$\alpha^{\leq n} = \top? \cup \alpha \cup (\alpha; \alpha) \cup \ldots \cup (\underbrace{\alpha; \ldots; \alpha}_{n}).$$

In other words, $\alpha(n)$ results from α by replacing every occurrence of an action term of the form γ^* (which is not in the scope of a test $\psi?$) with $\gamma^{\leq n}$. In particular, $\alpha(n)$ contains no occurrence of * and we can compute the length of $\alpha(n)$.

Finally, we put

$$l(\varphi) = \max\{|\alpha(b(\varphi) \cdot p(\varphi))| : [\alpha]\psi \in sub(\varphi)\}$$

We are in a position now to formulate and prove the main result of the paper.

THEOREM 5.3 (satisfiability criterion) *A formula φ is satisfiable iff there is a satisfying set for φ each block in which contains at most*

$$N = 2 \cdot l(\varphi) \cdot p(\varphi) \cdot |sub(\varphi)|$$

states.

The decidability od **PDLK** follows immediately since (1) the number of blocks for φ with $\leq N$ states is (effectively) bounded, (2) it can be checked effectively whether a basic structure $\langle \mathcal{F}, q \rangle$ is a block, and (3) it can be checked effectively whether a finite set of blocks for φ is a satisfying set for φ.

For the proof we require the following concepts and one Lemma. Given a quasimodel $\langle \mathcal{F}, q \rangle$ and a run r in it we define for any α occuring in $sub(\varphi)$ the set $Path_r(\alpha)$ inductively as follows:

$$
\begin{array}{rcl}
Path_r(\alpha_i) & = & \{\langle w, \alpha_i, v\rangle : wT_{\alpha_i}(r)v\} \\
Path_r(\alpha_i^-) & = & \{\langle w, \alpha_i^-, v\rangle : wT_{\alpha_i^-}(r)v\} \\
Path_r(\alpha \cup \beta) & = & Path_r(\alpha) \cup Path_r(\beta) \\
Path_r(\alpha;\beta) & = & \{\langle w, \ldots, v, \ldots, u\rangle : \\
 & & \langle w, \ldots, v\rangle \in Path_r(\alpha), \langle v, \ldots, u\rangle \in Path_r(\beta)\} \\
Path_r(\alpha^*) & = & \{\langle w\rangle : w \in W\} \\
 & & \bigcup\{Path_r(\alpha^n) : n > 0\} \\
Path_r(\psi?) & = & \{\langle w\rangle : \psi \in l_w(r(w))\}
\end{array}
$$

A path of the form $\langle w\rangle$ is called degenerate.

Observe that for any action term α without iteration the length of all sequences we obtain by deleting all action terms in sequences $\vec{w} \in Path_r(\alpha)$ is bounded by the length $|\alpha|$ of α.

For a path $\vec{w} = \langle w_0, \beta_0, \ldots, \beta_{k-1}, w_k\rangle$ we put $start(\vec{w}) = w_0$ and $end(\vec{w}) = w_k$. For two $\vec{v_1} = \langle w_0, \beta_0, \ldots, \beta_{k-1}, w_k\rangle$ and $\vec{v_2} = \langle w_k, \beta_k, \ldots, \beta_{n-1}, w_n\rangle$ we put

$$\vec{v_1} * \vec{v_2} = \langle w_0, \beta_0, \ldots, \beta_{k-1}, w_k, \beta_k, \ldots, \beta_{n-1}, w_n\rangle.$$

The following technical Lemma states that any state in a quasimodel can be duplicated without changing anything else. The easy proof is omitted.

LEMMA 5.4 (Duplication) *Let $\langle \mathcal{F}, q\rangle$ be a quasimodel for φ and $w \in W$, $w' \notin W$. Define $\langle \mathcal{F}', q'\rangle$ by putting $W' = W \cup \{w'\}$,*

$$q'(v) = \begin{cases} q(v) & : \; v \in W \\ q(w) & : \; v = w' \end{cases}$$

$$v_1 T'_{\alpha_i} v_2 \Leftrightarrow \begin{cases} v_1 T_{\alpha_i} v_2 & : \; v_1, v_2 \in W \\ w T_{\alpha_i} v_2 & : \; v_1 = w',\ v_2 \in W \\ v_1 T_{\alpha_i} w & : \; v_2 = w',\ v_1 \in W \\ w T_{\alpha_i} w & : \; w' = v_1 = v_2 \end{cases}$$

Then $\langle \mathcal{F}', q'\rangle$ is a quasimodel.

Proof of Theorem 5.3. Suppose φ is satisfiable. By Theorem 4.8 there is a quasimodel $\langle \mathcal{F}, q\rangle$ satisfying φ and having quasistates of depth $\leq d(\varphi)$. Let $\mathcal{R} = \langle \mathcal{R}_0, \ldots, \mathcal{R}_m\rangle$ be a suitable sequence for completely saturated runs in $\langle \mathcal{F}, q\rangle$ and assume that $\mathcal{F} = \langle W, T_{\alpha_1}, \ldots, T_{\alpha_n}\rangle$.

To construct a satisfying set $\mathcal{S}$ we associate with each quasistate $w \in W$ a block

$$\langle \mathcal{G}_w, q_w \rangle.$$

Fix $w_0 \in W$. We are going to construct $\langle \mathcal{G}_{w_0}, q_{w_0} \rangle$. Firstly we will obtain a block $\langle \mathcal{G}_0, q_0 \rangle$ the paths in which are too long which will then be modified to define the required block $\langle \mathcal{G}_{w_0}, q_{w_0} \rangle$.

Define sets $\mathcal{Q}_k$, $0 \leq k \leq m$, of runs by induction:

- $\mathcal{Q}_0 = \{r_0\}$,
- $\mathcal{Q}_{k+1}$ is constructed as follows: For any run $r \in \mathcal{Q}_k$ and $x \in T_{w_0}$ with $r(w_0) <_{w_0} x$ put an $r' \in \mathcal{R}_{k+1}$ with $r < r'$ and $r'(w_0) = x$ into $\mathcal{Q}_{k+1}$.

For any $r \in \mathcal{Q}_k$, $0 \leq k \leq m$ and any $[\alpha]\psi \in sub(\varphi)$ with $[\alpha]\psi \notin l_{w_0}(r(w_0))$ such that there exists a non–degenerate path

$$\langle w_0, \beta_0, w_1 \ldots, \beta_{m-1}, w_m \rangle \in Path_r(\alpha)$$

with $\psi \notin l_{w_m}(r(w_m))$ select two paths, $\vec{w_1}(r, [\alpha]\psi)$ and $\vec{w_2}(r, [\alpha]\psi)$, with this property. Observe that we may assume that there are two such paths by duplicating certain points in W, see Lemma 5.4. Moreover, we may assume that $\vec{w_i}(r, [\alpha]\psi) \neq \vec{w_j}(r', [\alpha']\psi')$, whenever $\langle r, [\alpha]\psi \rangle \neq \langle r', [\alpha']\psi' \rangle$ and $i, j \in \{1, 2\}$. Denote by SEL the set of all selected paths. Again by duplicating certain elements of W we may assume that no w_i occurs more than once in a path from SEL and that no w_i, $i \neq 0$ occurs in two different paths from SEL.

The cardinality of SEL is bounded by $2 \cdot p(\varphi) \cdot |sub(\varphi)|$.

Define $\langle \mathcal{G}_0, q_0 \rangle$ by putting $w \in W_0$ iff $w = w_0$ or there exists $\vec{w} \in SEL$ such that $\vec{w} = \vec{v_1} * \vec{v_2}$ and $w = end(\vec{v_1})$. For $w_1, w_2 \in W_0$ let $w_1 T'_{\alpha_i} w_2$ iff there exists $\vec{v} \in SEL$ such that

$$\vec{v} = \langle w_0, \ldots, w_1, \alpha_i, w_2, \ldots, w_m \rangle.$$

Finally define, for $w \in W_0$, $q_0(w) = q(w)$ and let $\mathcal{G}_0 = \langle W_0, T'_{\alpha_1}, \ldots, T'_{\alpha_n} \rangle$.

Clearly $\langle \mathcal{G}_0, q_0 \rangle$ satisfies the conditions (B1) ... (B4) of the definition of a block. We now verify that there exists a sequence $\langle \mathcal{P}_0, \ldots, \mathcal{P}_m \rangle$ in $\langle \mathcal{G}_0, q_0 \rangle$ which is suitable for weakly and root saturated runs in $\langle \mathcal{G}_0, q_0 \rangle$. Observe that the restriction of any $r \in \mathcal{Q}_k$ to W_0 is a weakly and root saturated run in $\langle \mathcal{G}_0, q_0 \rangle$ and that the restriction of any $r \in \mathcal{R}_k$ to W_0 is a weakly saturated run in $\langle \mathcal{G}_0, q_0 \rangle$. However, not all those restrictions are root saturated and so we have to construct certain runs.

For any $k \leq m$, $\vec{w} \in SEL$, $r \in \mathcal{R}_k$, and any $r' \in \mathcal{Q}_k$ with $r'(w_0) = r(w_0)$ we put into $\mathcal{P}_k$ the run $r +_{\vec{w}} r'$, defined by taking for $v \in W_0$

$$r +_{\vec{w}} r'(v) = \begin{cases} r(v) & : \quad v \text{ occurs in } \vec{w} \\ r'(v) & : \quad \text{otherwise.} \end{cases}$$

(Notice that at least one such r' exists, since for any $x \in T_{w_0}$ we selected a run through x.) We show that $r +_{\vec{w}} r'$ is a weakly and root saturated run in $\langle \mathcal{G}_0, q_0 \rangle$. Clearly $r +_{\vec{w}} r'$ is a weakly saturated run. Suppose that $[\alpha]\psi \notin l_{w_0}(r +_{\vec{w}} r'(w_0))$. If $\langle w_0 \rangle \in Path_{r'}(\alpha)$ and $\psi \notin l_{w_0}(r'(w_0))$, then there is nothing to show. Otherwise there exists a non–degenerate $\vec{v} \in Path_{r'}(\alpha)$ from SEL such that $\psi \notin l_{end(\vec{v})}(r'(end(\vec{v}))$ and $\vec{v} \neq \vec{w}$. (Here we use the fact that we took two paths $\vec{w}_i(r', [\alpha]\psi)$, $i = 1, 2$, to saturate $[\alpha]\psi$ with respect to r'.) Then $end(\vec{v})$ saturates $[\alpha]\psi$ with respect to $r +_{\vec{w}} r'$.

All $\mathcal{P}_k$ contain only weakly and root saturated runs in $\langle \mathcal{G}_0, q_0 \rangle$. We now show that the sequence $\mathcal{P} = \langle \mathcal{P}_0, \ldots, \mathcal{P}_m \rangle$ is suitable: (S1) $r_0 \in \mathcal{Q}_0$ and so $r_0 \in \mathcal{P}_0$. (S2) Let $r +_{\vec{w}} r' \in \mathcal{P}_{k+1}$. Take $s \in \mathcal{R}_k$ with $s < r$ and $s' \in \mathcal{Q}_k$ with $s' < r'$. Then $s +_{\vec{w}} s' < r +_{\vec{w}} r'$ and $s +_{\vec{w}} s' \in \mathcal{P}_k$. (S3) Let $r +_{\vec{w}} r' \in \mathcal{P}_k$, $v \in W_0$, and $x \in T_v$ such that $r +_{\vec{w}} r'(v) <_v x$.

Case 1. v occurs in $\vec{w}$. Take $s \in \mathcal{R}_{k+1}$ such that $r < s$ and $s(v) = x$. Take $s' \in \mathcal{Q}_{k+1}$ such that $s'(w_0) = s(w_0)$ and such that $r' < s'$. Then $r +_{\vec{w}} r' < s +_{\vec{w}} s' \in \mathcal{P}_{k+1}$ and $s +_{\vec{w}} s'(v) = x$.

Case 2. v does not occur in $\vec{w}$. Take $s' \in \mathcal{Q}_{k+1}$ such that $r' < s'$ and $s'(v) = x$. Take $s \in \mathcal{R}_{k+1}$ such that $s(w_0) = s'(w_0)$ and such that $r < s$. Then $r +_{\vec{w}} r' < s +_{\vec{w}} s' \in \mathcal{P}_{k+1}$ and $s +_{\vec{w}} s'(v) = x$.

It is shown that $\langle \mathcal{G}_0, q_0 \rangle$ is a block. Our next step is to extract from $\langle \mathcal{G}_0, q_0 \rangle$ a substructure $\langle \mathcal{G}_{w_0}, q_{w_0} \rangle$ which is still a block with root w_0 but the length of branches in which are bounded by $l(\varphi)$.

Consider a sequence $\vec{w} \in SEL$ which was selected to saturate $[\alpha]\psi$ with respect to a run r (i.e., $\vec{w} = \vec{w}_i(r, [\alpha]\psi)$, for an $i \in \{1, 2\}$). We define a truncated version $tr^r_\alpha(\vec{w})$ of $\vec{w}$.

$tr^r_\alpha(\vec{w})$ is defined by induction on the complexity of α. If α does not contain an occurence of * then, since $\vec{w} \in Path_r(\alpha)$, the length of $\vec{w}$ is at most $|\alpha|$. In this case we put $tr^r_\alpha(\vec{w}) = \vec{w}$.

Suppose now that α contains iteration. If $\alpha = \beta \cup \gamma$, then

$$tr^r_\alpha(\vec{w}) = \begin{cases} tr^r_\beta(\vec{w}) & : \quad \vec{w} \in Path_r(\beta) \\ tr^r_\gamma(\vec{w}) & : \quad \text{otherwise} \end{cases}$$

If $\alpha = \beta; \gamma$, then $\vec{w} = \vec{w_1} * \vec{w_2}$ with $\vec{w_1} \in Path_r(\beta)$ and $\vec{w_2} \in Path_r(\gamma)$ and we put

$$tr^r_\alpha(\vec{w}) = tr^r_\beta(\vec{w_1}) * tr^r_\gamma(\vec{w_2}).$$

Let $\alpha = \beta^*$. Then there exists $n \in \omega$ and $\vec{w_1}, \ldots \vec{w_n}$ such that $\vec{w_i} \in Path_r(\beta)$ and

$$\vec{w} = \vec{w_1} * \ldots * \vec{w_n}.$$

If $n \leq b(\varphi) \cdot p(\varphi)$, then we put

$$tr^r_\alpha(\vec{w}) = tr^r_\beta(\vec{w_1}) * \ldots * tr^r_\beta(\vec{w_n}).$$

Otherwise there must be $i, j \leq n$ with $i + 1 < j$ such that we have for $v_0 = end(\vec{w_i})$ and $v_1 = start(\vec{w_j})$:

- $q(v_0) = q(v_1)$ and $r(v_0) = r(v_1)$.

In this case we put

$$tr^r_\alpha(\vec{w}) = tr^r_\beta(\vec{w_1}) * \ldots * tr^r_\beta(\vec{w_i}) * tr^r_\beta(\vec{w_j}) * \ldots * tr^r_\beta(\vec{w_n}).$$

Finally, if $\alpha = \psi?$ then $tr^r_\alpha(\vec{w}) = \vec{w}$.

It should be clear that in this way we obtain a path $tr^r_\alpha(\vec{w})$ the length of which is bounded by $l(\varphi)$. Denote by SEL' the set of all $tr^r_\alpha(\vec{w})$, $\vec{w}$ was put into SEL to saturate an $[\alpha]\psi \in sub(\varphi)$ with respect to a run $r \in \mathcal{Q}_k$, $0 \leq k \leq m$. We construct from SEL' the required block $\langle \mathcal{G}_{w_0}, q_{w_0}\rangle$:

Let $w \in W_{w_0}$ iff $w = w_0$ or there exists $\vec{w} \in SEL'$ such that $\vec{w} = \vec{v_1} * \vec{v_2}$ and $w = end(\vec{v_1})$. For $w_1, w_2 \in W_{w_0}$ put $w_1 S_{\alpha_i} w_2$ iff there exists $\vec{v} \in SEL'$ such that

$$\vec{v} = \langle w_0, \ldots, w_1, \alpha_i, w_2, \ldots, w_m\rangle.$$

Finally define, for $w \in W_{w_0}$, $q_{w_0}(w) = q(w)$ and let $\mathcal{G}_{w_0} = \langle W_{w_0}, S_{\alpha_1}, \ldots, S_{\alpha_n}\rangle$. It is easy but (extremely) tedious to show that $\langle \mathcal{G}_{w_0}, q_{w_0}\rangle$ is a block. We leave this to the reader.

The required satisfying set $\mathcal{S}$ for φ is the set of all blocks $\langle \mathcal{G}_w, q_w\rangle$, $w \in W$.

Now let $\mathcal{S}$ be a satisfying set of φ. We are going to construct a quasimodel $\langle \mathcal{F}, q\rangle$ as the limit of an inductively defined sequence of weak quasimodels

$$\langle\langle \mathcal{F}_m, q_m\rangle : m \in \omega\rangle.$$

Let $\langle \mathcal{F}_0, q_0\rangle$ be a block with root w_0 such that $\varphi \in l_{w_0}(r_0(w_0))$. Suppose now that we have constructed $\langle \mathcal{F}_m, q_m\rangle$ with $\mathcal{F}_m = \langle W_m, T^m_{\alpha_1}, \ldots, T^m_{\alpha_n}\rangle$ already. For every $w \in W_m - W_{m-1}$ (here and in what follows $W_{-1} = \{w_0\}$) select a block $\langle \mathcal{G}_w, q_w\rangle$ with root w such that $q_w(w) = q_m(w)$. Let $\mathcal{G}_w = \langle W_w, T^w_{\alpha_1}, \ldots, T^w_{\alpha_n}\rangle$. We may assume that all the selected blocks are mutually disjoint and that $W_w \cap W_m = \{w\}$, for $w \in W_m - W_{m-1}$. Define $\langle \mathcal{F}_{m+1}, q_{m+1}\rangle$ by putting

$$W_{m+1} = W_m \cup \bigcup\{W_w : w \in W_m - W_{m-1}\},$$

$$T^{m+1}_{\alpha_i} = T^m_{\alpha_i} \cup \bigcup\{T^w_{\alpha_i} : w \in W_m - W_{m-1}\},$$

$$q_{m+1}(v) = \begin{cases} q_m(v) & : \quad v \in W_m \\ q_w(v) & : \quad v \in W_w - W_m \end{cases}$$

The limit $\langle \mathcal{F} = \langle W, T_{\alpha_1}, \ldots T_{\alpha_n}\rangle, q\rangle$ is defined by putting

$$W = \bigcup\{W_m : m \in \omega\}, \;\; T_{\alpha_i} = \bigcup\{T^m_{\alpha_i} : m \in \omega\},$$

$$q(w) = \bigcup\{q_m(w) : \; m \in \omega, \; w \in W_m\}.$$

It remains to show that $\langle \mathcal{F}, q\rangle$ is a quasimodel for φ. To this end we construct a sequence $\mathcal{R} = \langle \mathcal{R}_0, \ldots, \mathcal{R}_m\rangle$ which is suitable for completely saturated runs. For any $\langle \mathcal{G}_w, q_w \rangle \in \mathcal{S}$ which was used to construct $\mathcal{F}$ take a sequence $\mathcal{R}^w = \langle \mathcal{R}_0^w, \ldots, \mathcal{R}_m^w \rangle$ which is suitable for weakly and root saturated runs in $\langle \mathcal{G}_w, q_w \rangle$. Take k with $0 \leq k \leq m$. Then $\mathcal{R}_k$ consists of all runs r which can be constructed inductively by putting $r = \bigcup \{ r^m : m \in \omega \}$, where the r^m are k–runs in $\langle \mathcal{F}_m, q_m \rangle$, for $m \in \omega$, constructed in the following manner: take $r^0 \in \mathcal{R}_k^{w_0}$ and suppose that r^m has been defined. For any $w \in W_m - W_{m-1}$ take $r_w \in \mathcal{R}_k^w$ such that $r^m(w) = r_w(w)$ and put for $v \in W_{m+1}$

$$r^{m+1}(v) = \begin{cases} r^m(v) & : \quad v \in W_m \\ r_w(v) & : \quad v \in W_w - W_m \end{cases}$$

Observe that $\mathcal{R}_k$ contains completely saturated runs only: that all $r \in \mathcal{R}_k$ are weakly saturated can be proved by induction and is left to the reader. For the other direction let $r = \bigcup \{ r^m : m \in \omega \} \in \mathcal{R}_k$, $w \in W$ and suppose that $[\alpha]\psi \notin l_w(r(w))$. Then $\langle \mathcal{G}_w, q_w \rangle \in \mathcal{S}$ is a substructure of $\langle \mathcal{F}, q \rangle$ such that there exists a weakly and rooted saturated run r_w in $\langle \mathcal{G}_w, q_w \rangle$ which coincides with the restriction of r to W_w. $\langle \mathcal{G}_w, q_w \rangle$ is a block, and so there exists $v \in W_w$ such that $wT_\alpha^w(r_w)v$ with $\psi \notin l_v(r_w(v))$. Hence $wT_\alpha(r)v$ and $\psi \notin l_v(r(v))$.

That $\mathcal{R}$ is suitable follows immediately from the construction. ❑

6 Related Logics

In this section we shall discuss decidability results for products which either follow directly from the decidability of $\mathbf{PDLK}_m$ or can be proved by similar methods. We introduce some notation first: Given two propositional modal languages $\mathcal{L}_1$ and $\mathcal{L}_2$ interpreted in frames of the form $\mathcal{F} = \langle W, S_1, \ldots, S_n \rangle$ and $\mathcal{G} = \langle V, R_1, \ldots, R_m \rangle$, respectively, we form the language $\mathcal{L}_1 \oplus \mathcal{L}_2$ as follows:

- $\mathcal{L}_1 \oplus \mathcal{L}_2$ contains an infinite set of propositional variables and is closed under the booleans: $\wedge$, $\neg$.
- for any $i \in \{1, 2\}$ and modal operator f of $\mathcal{L}_i$ of arity k the formula $f(\varphi_1, \ldots, \varphi_k)$ is in $\mathcal{L}_1 \oplus \mathcal{L}_2$ whenever $\varphi_1, \ldots, \varphi_k$ are in $\mathcal{L}_1 \oplus \mathcal{L}_2$.

Formulas from $\mathcal{L}_1 \oplus \mathcal{L}_2$ are interpreted in products

$$\mathcal{F} \times \mathcal{G} = \langle W \times V, \underline{R_1}, \ldots, \underline{R_n}, \underline{S_1}, \ldots, \underline{S_m} \rangle$$

in the obvious manner. Given two modal logics L_1 and L_2 in languages $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, we denote by $L_1 \times L_2$ the set of all formulas in $\mathcal{L}_1 \oplus \mathcal{L}_2$ which are valid in all members of

$$\{ \mathcal{F} \times \mathcal{G} : \mathcal{F} \models L_1, \ \mathcal{G} \models L_2 \}.$$

(We write $\mathcal{F} \models L$ whenever all formulas in L are valid in $\mathcal{F}$).

We are in the position now to formulate various decidability results of interest. By **S4** we denote the monomodal logic determined by the class of transitive and reflexive frames, and by **K4** the logic determined by the class of transitive frames. By **K4t** and **S4t** we denote the bimodal logics of the classes of frames $\langle W, R, R^{-1}\rangle$, where R transitive and R is reflexive and transitive, respectively.

THEOREM 6.1 *The logics* $\mathbf{S4} \times \mathbf{K}_m$*,* $\mathbf{S4t} \times \mathbf{K}_m$*,* $\mathbf{K4} \times \mathbf{K}_m$*, and* $\mathbf{K4t} \times \mathbf{K}_m$ *are decidable.*

Proof $\mathbf{S4} \times \mathbf{K}_m$ as well as $\mathbf{S4t} \times \mathbf{K}_m$ are embedded in $\mathbf{PDLK}_m$ by means of the translation which replaces the modal operator $\Box$ of **S4** with $[\alpha_1^*]$. So they are decidable. The translation which replaces the modal operator $\Box$ of **K4** with $[\alpha_1][\alpha_1^*]$ gives an embedding of $\mathbf{K4} \times \mathbf{K}_m$ into $\mathbf{PDLK}_m$. ❑

By $\mathcal{L}_{\mathbf{CS5}_n}$ we denote the propositional modal language with epistemic operators K_M, for each non-empty set $M \subseteq \{1, \ldots, n\}$. This language is interpreted in structures of the form

$$\langle W, \langle R_M : M \subseteq \{1, \ldots, n\}, M \neq \emptyset\rangle\rangle$$

such that each $R_{\{i\}}$ is an equivalence relation on W and R_M is the reflexive and transitive closure of $\bigcup\{R_{\{i\}} : i \in M\}$, $|M| > 1$. Denote by $\mathbf{S5}_n^C$ the set of all valid formulas. This logic is one of the most important modal logics of knowledge, cf. Halpern and Moses (1992) and Fagin et al. (1995). The operators $\mathsf{K}_{\{i\}}$ can be interpreted as "*agent i knows*" while the operators K_M, $|M| > 1$, can be interpreted as "*it is common knowledge among the agents in* M". We now show that $\mathbf{S5}_n^C \times \mathbf{K}_m$ can be embedded into $\mathbf{PDLK}_m$.

Define the following translation T from $\mathcal{L}_{\mathbf{CS5}_n} \oplus \mathcal{L}_{\mathbf{K}_m}$ into $\mathcal{L}_{\mathbf{PDLK}_m}$ by putting

$$\begin{aligned}
T(p_i) &= p_i, \\
T(\varphi \wedge \psi) &= T(\varphi) \wedge T(\psi), \\
T(\neg\varphi) &= \neg T(\varphi), \\
T(\Box_i \varphi) &= \Box_i T(\varphi), \\
T(\mathsf{K}_M \varphi) &= [(\alpha_{i_1} \cup \alpha_{i_1}^- \cup \ldots \cup \alpha_{i_k} \cup \alpha_{i_k}^-)^*]\, T(\varphi), \text{ for } M = \{i_1, \ldots i_k\}.
\end{aligned}$$

Observe that T is similar to the translation defined by Fischer and Immerman (1987).

THEOREM 6.2 *For all* $\varphi \in \mathcal{L}_{\mathbf{CS5}_n} \oplus \mathcal{L}_{\mathbf{K}_m}$*,*

$$\varphi \in \mathbf{S5}_n^C \times \mathbf{K}_m \Leftrightarrow T(\varphi) \in \mathbf{PDLK}_m.$$

Hence $\mathbf{S5}_n^C \times \mathbf{K}_m$ *is decidable.*

The proof is easy and omitted.

We finally mention decidability results for products of temporal logics with $\mathbf{K}_m$ which do not immediately follow from the decidability of $\mathbf{PDLK}_m$ but which can be proved by combining in a straightforward manner the notion of a quasimodel introduced in the present paper with the technique introduced in Wolter and Zakharyaschev (1998c).

Denote by **Lin** the bimodal logic determined by the class of frames $\langle W, R, R^{-1}\rangle$ such that R is a strict linear ordering and denote by $\mathbf{L}_Q$ the bimodal logic determined by the frame $\langle \mathbf{Q}, <, <^{-1}\rangle$, where $\mathbf{Q}$ denotes the set of rational numbers.

THEOREM 6.3 *The logics* $\mathbf{Lin} \times \mathbf{K}_m$ *and* $\mathbf{L}_Q \times \mathbf{K}_m$ *are decidable.*

We omit the proof: as mentioned above, it is a straightforward combination of our notion of a quasimodel and the mosaic method developed in Wolter and Zakharyaschev (1998c), see also Reynolds (1996).

We come to products of temporal logics with the binary operators *Since* and *Until* and $\mathbf{K}_m$. Suppose that $\langle W, <\rangle$ is a strict linear ordering and $\mathcal{V}$ is a valuation of the propositional variables into 2^W. The truth conditions for the operators *Since* and *Until* are defined as follows:

- $w \models \varphi$ *Since* ψ iff there exists $w' < w$ with $w' \models \psi$ and for all v such that $w' < v < w$ we have $v \models \varphi$.
- $w \models \varphi$ *Until* ψ iff there exists $w' > w$ with $w' \models \psi$ and for all v such that $w < v < w'$ we have $v \models \varphi$.

The set of formulas which are valid in a strict linear ordering $\langle W, <\rangle$ is denoted by $\mathbf{TP}_{\langle W,<\rangle}$. Temporal logics of this type have been investigated intensively, we refer the reader to Gabbay et al. (1994) for a comprehensive survey.

THEOREM 6.4 *The logics* $\mathbf{TP}_{\langle \mathbf{N},<\rangle} \times \mathbf{K}_m$ *(*$\mathbf{N}$ *the set of natural numbers) and* $\mathbf{TP}_{\langle \mathbf{Z},<\rangle} \times \mathbf{K}_m$ *(*$\mathbf{Z}$ *the set of integers) are decidable.*

Again the proof is omitted because it is a straightforward combination of the notion of a quasimodel intoduced here, and the technique developed in Wolter and Zakharyaschev (1998c).

7 Modal description logics

In this section we translate the results of the previous sections to modal description logics, and show that decidability is preserved when local roles names are added to the resulting language.

The alphabet of the description logic $\mathcal{ALC}$ consists of an infinite set of concept names $C_1, \ldots$ and an infinite set of role names $R_1, \ldots$. Complex concepts are defined inductively: if C and D are concepts and R is a role name, then

$C \wedge D$, $\neg C$, and $\exists R.C$ are concepts. For example, let child and $\mathsf{employed}$ be role names, and male, female, and $\mathsf{company}$ be concept names. Then we can form

$$\mathsf{male} \wedge \exists \mathsf{child}.(\mathsf{female} \wedge \exists \mathsf{employed}.\mathsf{company}),$$

i.e., the concept comprising all males with a daughter who is employed by a company. In $\mathcal{ALC}$ only static and time–independent knowledge can be represented. Suppose, however, that we want to represent dynamic or temporal knowledge, say the concept comprising all males who have a daughter who, whenever she passes the examination, is employed by a company. By extending $\mathcal{ALC}$ by the action variable $\mathsf{pass\ exam}$ this can be expressed as as follows:

$$\mathsf{male} \wedge \exists \mathsf{child}.(\mathsf{female} \wedge [\mathsf{pass\ \ exam}]\exists \mathsf{employed}.\mathsf{company}).$$

In this section we shall investigate the extension of $\mathcal{ALC}$ by converse **PDL**. Extensions of $\mathcal{ALC}$ by temporal logics or modal logics of knowledge can be treated in the same way.

DEFINITION 7.1 (Alphabet) The language **CPDLC** has the following alphabet:

- a set of concept names: $C_0, C_1, \ldots,$
- the booleans: $\wedge$, $\neg$, $\top$,
- a set of global role names: $R_1, \ldots,$
- a set of local role names: $S_1, \ldots,$
- action variables: $\alpha_1, \alpha_2 \ldots,$
- action term constructors: $\cup$ (alternation), ; (composition), * (iteration), ? (test), $^-$ (converse).

DEFINITION 7.2 (Concept, action term) The definition of action terms is as before, save that now $C?$ is an action term for any concepts C. Concepts are defined as follows

- any concept name is a concept.
- if C_1, C_2 are concepts, then $\neg C_1$ and $C_1 \wedge C_2$ are concepts.
- if R is a (local or global) role name and C a concept, then $\exists R.C$ is a concept.
- if C is a concept and α an action term, then $[\alpha]C$ is a concept.

We obtain models for this language as follows: suppose that $\mathcal{F} = \langle W, T_{\alpha_1}, \ldots \rangle$ is a **PDL**–structure, Δ is a set of objects, and that $\mathcal{M}$ is a function associating with every $w \in W$ an $\mathcal{ALC}$–model

$$\mathcal{M}(w) = \langle \Delta, C_1^{\mathcal{M}(w)}, \ldots, R_1^{\mathcal{M}(w)}, \ldots, S_1^{\mathcal{M}(w)}, \ldots \rangle,$$

where the $C_i^{\mathcal{M}(w)}$ are subsets of Δ (interpreting the concept names), the $R_i^{\mathcal{M}(w)}$ are binary relations on Δ (interpreting the global role names) such that $R_i^{\mathcal{M}(w)} = R_i^{\mathcal{M}(v)}$, for all $w, v \in W$, and the $S_i^{\mathcal{M}(w)}$ are binary relations on Δ (interpreting the local role names). So the interpretation of global role names does not depend on the state–dimension while the interpretation of local roles names is allowed to be different in different states. Then $\langle \mathcal{F}, \mathcal{M} \rangle$ is called a **CPDLC**–model. For any $w \in W$ and concept C the set $C^{\mathcal{M}(w)} \subseteq \Delta$ and the relation $T_\alpha^{\mathcal{M}}$, α an action term, are defined as follows:

- $C^{\mathcal{M}(w)} = C_i^{\mathcal{M}(w)}$, whenever $C = C_i$,
- $(C \wedge D)^{\mathcal{M}(w)} = C^{\mathcal{M}(w)} \cap C^{\mathcal{M}(w)}$,
- $(\neg C)^{\mathcal{M}(w)} = \Delta - C^{\mathcal{M}(w)}$,
- $(\exists R.C)^{\mathcal{M}(w)} = \{x \in \Delta : \exists y \in \Delta(x R^{\mathcal{M}(w)} y \wedge y \in C^{\mathcal{M}(w)})\}$,
- $([\alpha]C)^{\mathcal{M}(w)} = \{x \in \Delta : \forall v \in W \; (\langle w, x \rangle T_\alpha^{\mathcal{M}} \langle v, x \rangle \Rightarrow x \in C^{\mathcal{M}(v)})$.
- $T_{\alpha_i}^{\mathcal{M}} = \{\langle \langle w, x \rangle, \langle v, x \rangle \rangle : w T_{\alpha_i} v\}$.
- $T_{\alpha;\beta}^{\mathcal{M}} = T_\alpha^{\mathcal{M}} \circ T_\beta^{\mathcal{M}}$.
- $T_{\alpha \cup \beta}^{\mathcal{M}} = T_\alpha^{\mathcal{M}} \cup T_\beta^{\mathcal{M}}$.
- $T_{\alpha^*}^{\mathcal{M}} = (T_\alpha^{\mathcal{M}})^*$.
- $T_{\alpha_i^-}^{\mathcal{M}} = (T_{\alpha_i}^{\mathcal{M}})^{-1}$.
- $T_{C?}^{\mathcal{M}} = \{\langle \langle w, x \rangle, \langle w, x \rangle \rangle : \; x \in C^{\mathcal{M}(w)}\}$.

A concept C is satisfiable iff there exists a model $\langle \mathcal{F}, \mathcal{M} \rangle$ and a state w in it such that $C^{\mathcal{M}(w)} \neq \emptyset$. Obviously **CPDLC** without local role names is just a syntactic variant of $\mathbf{PDLK}_m$[3]. So we obtain

COROLLARY 7.3 *It is decidable whether a concept C without local role names is satisfiable in a* **CPDLC***–model.*

[3]**CPDLC** has infinitely many role names, and so we require infinitely many modal operators to define a translation from the language of **CPDLC** into the modal language. But obviously everything proved for $\mathcal{L}_{\mathbf{PDLK}_m}$ holds for $\mathcal{L}_{\mathbf{PDLK}_\omega}$, the language with infinitely many modal operators $\Box_1, \ldots$

Consequently, the only new ingredients of **CPDLC** are the local role names. Observe that even for the simple example given above, local role names are required: the interpretation of the role employed should certainly depend on the states. (On the other hand, the interpretation of the role child should not depend on the states, and thus be a global role name.)

We are now going to prove the decidability for the full language **CPDLC**. This will be achieved by simulating local roles by means of global ones and certain concepts.

For any local role name S take a new concept name reach_S and a new global role name R_S. Define the following translation σ from the set of arbitrary concepts and action terms into the set of concepts and action terms containing global roles only:

$$\begin{array}{rcl}
\sigma(C_i) & = & C_i, \\
\sigma(C \wedge D) & = & \sigma(C) \wedge \sigma(D), \\
\sigma(\neg C) & = & \neg\sigma(C), \\
\sigma([\alpha]C) & = & [\sigma(\alpha)]\sigma(C), \\
\sigma(\exists R.C) & = & \exists R.\sigma(C),\ R \text{ is a global role name}, \\
\sigma(\exists S.C) & = & \exists R_S.(\mathsf{reach}_S \wedge \sigma(C)),\ S \text{ is a local role name}, \\
\sigma(\alpha_i) & = & \alpha_i, \\
\sigma(\alpha^f) & = & (\sigma(\alpha))^f,\ \text{for } f \in \{\,^-,\,^*\}, \\
\sigma(\alpha g \beta) & = & \sigma(\alpha) g \sigma(\beta),\ \text{for } g \in \{\circ, \cup\}, \\
\sigma(C?) & = & \sigma(C)?.
\end{array}$$

THEOREM 7.4 *A concept C is satisfiable in a* **CPDLC***–model iff $\sigma(C)$ is satisfiable in a* **CPDLC***–model. It is decidable whether a concept C is satisfiable in a* **CPDLC***–model.*

To prove this Theorem we require the following Lemma.

LEMMA 7.5 *If a concept C is satisfiable, then it is satisfiable in a model $\langle \mathcal{F}, \mathcal{M} \rangle$ such that for any $x \in \Delta$ and (global or local) role name R:*

$$|\{y \in \Delta : \exists w \in W (y R^{\mathcal{M}(w)} x)\}| \leq 1.$$

Proof Suppose that C is satisfiable in a model $\langle \mathcal{F}, \mathcal{M} \rangle$. Assume $x_0 \in C^{\mathcal{M}(w)}$, $\mathcal{F} = \langle W, T_{\alpha_1}, \ldots, T_{\alpha_n} \rangle$, and

$$\mathcal{M}(w) = \langle \Delta, C_1^{\mathcal{M}(w)}, \ldots, R_1^{\mathcal{M}(w)}, \ldots, S_1^{\mathcal{M}(w)}, \ldots \rangle.$$

For each S_i let

$$R[S_i]^{\mathcal{M}} = \bigcup \{S_i^{\mathcal{M}(w)} : w \in W\}.$$

Put $Q_{2i} = R_i^{\mathcal{M}(w)}$ and $Q_{2i+1} = R[S_i]^{\mathcal{M}}$, for $i \in \omega$. We unravel the resulting model by putting for $w \in W$:

$$\mathcal{N}(w) = \langle \Delta', C_1^{\mathcal{N}(w)}, \ldots, R_1^{\mathcal{N}(w)}, \ldots, S_1^{\mathcal{N}(w)}, \ldots \rangle,$$

where

$$\Delta' = \{\langle x_0, Q_{i_1}, \ldots, Q_{i_m}, x_m \rangle : m \in \omega, \ \forall 1 \leq j \leq m \ (x_{j-1} Q_{i_j} x_j)\},$$

$R_i^{\mathcal{N}(w)}$ is defined by putting $\langle x_0, Q_{i_1}, \ldots, Q_{i_m}, x_m \rangle R_i^{\mathcal{N}(w)} x$ iff

$$\exists y \ \ (x = \langle x_0, Q_{i_1}, \ldots, Q_{i_m}, x_m, R_i^{\mathcal{M}(w)}, y \rangle),$$

$S_i^{\mathcal{N}(w)}$ is defined by putting $\langle x_0, Q_{i_1}, \ldots, Q_{i_m}, x_m \rangle S_i^{\mathcal{N}(w)} x$ iff

$$\exists y \ \ (x = \langle x_0, Q_{i_1}, \ldots, Q_{i_m}, x_m, R[S_i]^{\mathcal{M}}, y \rangle \text{ and } x_m S_i^{\mathcal{M}(w)} y),$$

and $C_i^{\mathcal{N}(w)}$ is defined by putting

$$\langle x_0, Q_{i_1}, \ldots, Q_{i_m}, x_m \rangle \in C_i^{\mathcal{N}(w)} \ \Leftrightarrow \ x_m \in C_i^{\mathcal{M}(w)}.$$

Obviously $\langle \mathcal{F}, \mathcal{N} \rangle$ satisfies the condition formulated in Lemma 7.5. By induction one can prove for every concept D:

$$\langle x_0, Q_{i_1}, \ldots, Q_{i_m}, x_m \rangle \in D^{\mathcal{N}(w)} \ \Leftrightarrow \ x_m \in D^{\mathcal{M}(w)}.$$

Hence $\langle x_0 \rangle \in C^{\mathcal{N}(w)}$. ❑

Proof of Theorem 7.4. It suffices to prove the first claim. Suppose first that C is satisfiable. Then C is satisfiable in a model $\langle \mathcal{F}, \mathcal{M} \rangle$ satisfying the condition formulated in Lemma 7.5. Let $\mathcal{F} = \langle W, T_{\alpha_1}, \ldots, T_{\alpha_n} \rangle$ and

$$\mathcal{M}(w) = \langle \Delta, C_1^{\mathcal{M}(w)}, \ldots, R_1^{\mathcal{M}(w)}, \ldots, S_1^{\mathcal{M}(w)}, \ldots \rangle.$$

We define a model $\langle \mathcal{F}, \mathcal{N} \rangle$ with

$$\mathcal{N}(w) = \langle \Delta, C_1^{\mathcal{N}(w)} \ldots, \mathsf{reach}_{S_1}^{\mathcal{N}(w)}, \ldots, R_1^{\mathcal{N}(w)}, \ldots, R_{S_1}^{\mathcal{N}(w)}, \ldots \rangle$$

by putting:

- for any concept name C_i not of the form reach_S:

$$C_i^{\mathcal{N}(w)} = C_i^{\mathcal{M}(w)},$$

- for any concept of the form reach_{S_i}:

$$\mathsf{reach}_{S_i}^{\mathcal{N}(w)} = \{x \in \Delta : \exists y \in \Delta \ (y S_i^{\mathcal{M}(w)} x)\},$$

- for any local role S_i:

$$R_{S_i}^{\mathcal{N}(w)} = \bigcup\{S_i^{\mathcal{M}(v)} : v \in W\},$$

- for any global role R_i:

$$R_i^{\mathcal{N}(w)} = R_i^{\mathcal{M}(w)}.$$

Now $D^{\mathcal{M}(w)} = (\sigma(D))^{\mathcal{N}(w)}$, for all concepts D and $w \in W$, follows by induction. We show the induction step for $C = \exists S_i.D$: suppose $x \in C^{\mathcal{M}(w)}$. Then there exists y with $xS_i^{\mathcal{M}(w)}y$ and $y \in D^{\mathcal{M}(w)}$. By induction hypothesis, $xR_{S_i}^{\mathcal{N}(w)}y$, $y \in \mathsf{reach}_{S_i}^{\mathcal{N}(w)}$, and $y \in \sigma(D)^{\mathcal{N}(w)}$. Hence $y \in (\exists R_{S_i}.(\mathsf{reach}_{S_i} \wedge \sigma(D)))^{\mathcal{N}(w)}$.

Conversely, suppose that $x \in \sigma(C)^{\mathcal{N}(w)}$. Then we find $y \in \mathsf{reach}_{S_i}^{\mathcal{N}(w)}$ with $xR_{S_i}^{\mathcal{N}(w)}y$ and $y \in \sigma(D)^{\mathcal{N}(w)}$. By the definition of $R_{S_i}^{\mathcal{N}(w)}$, there exists $v \in W$ such that $xS_i^{\mathcal{M}(v)}y$, and, by the definition of $\mathsf{reach}_{S_i}^{\mathcal{N}(w)}$, we find x' with $x'S_i^{\mathcal{M}(w)}y$. By the condition formulated in Lemma 7.5, $x = x'$ and so $xS_i^{\mathcal{M}(w)}y$. By induction hypothesis, $y \in D^{\mathcal{M}(w)}$, and so we obtain $x \in C^{\mathcal{M}(w)}$.

We come to the other direction. Suppose that $\sigma(C)$ is satisfiable in a model $\langle \mathcal{F}, \mathcal{M} \rangle$ of the form

$$\mathcal{M}(w) = \langle \Delta, C_1^{\mathcal{M}(w)}, \ldots, \mathsf{reach}_{S_1}^{\mathcal{M}(w)}, \ldots, R_1^{\mathcal{M}(w)}, \ldots, R_{S_1}^{\mathcal{M}(w)}, \ldots \rangle.$$

Define

$$\mathcal{N}(w) = \langle \Delta, C_1^{\mathcal{N}(w)}, \ldots, R_1^{\mathcal{N}(w)}, \ldots, S_1^{\mathcal{N}(w)}, \ldots \rangle$$

by leaving the interpretations of concepts C_i and global role names R_i unchanged and putting for any local role name S:

$$xS^{\mathcal{N}(w)}y \Leftrightarrow xR_S^{\mathcal{M}(w)}y \text{ and } y \in \mathsf{reach}_S^{\mathcal{M}(w)}.$$

Again, $D^{\mathcal{N}(w)} = (\sigma(D))^{\mathcal{M}(w)}$, for all concepts D and $w \in W$, follows by induction. ❑

We close this section with a brief comparison of the modal description logics introduced here with systems from the literature. We shall only indicate similarities and differences, a comprehensive classification of modal description logics is beyond the scope of this paper. The approaches which come closest to ours are the logics considered by Schild (1993) and Baader and Ohlbach (1995). The temporal description logic of Schild (1993) is actually the fragment of our temporal description logic without global role names. Baader and Ohlbach (1995) consider modal description logics with both local and global roles. However, they do not obtain decidability results. On the other hand, they consider systems which allow modal operators to be applied to roles: roles of the form $[\alpha]R$, R a role name, are interpreted as follows.

- $x[\alpha]R^{\mathcal{M}(w)}y$ iff $xR^{\mathcal{M}(v)}y$ holds for all v with $wT_\alpha v$.

Not much is known about the decidability of the satisfaction problem for languages with this (or similar) constructors. We shall see below, however, that the expansion of **CPDLC** by roles of the form $[\alpha]R$ has an undecidable satisfaction problem.

The approaches discussed so far do not take into account reasoning with axioms; that is to say, they do not allow the application of modal operators to equations $C = D$, where C and D are arbitrary concepts. Laux (1994) and Gräber et al. (1995) investigate modal description logics with modal operators applied to axioms but not to concepts. Those systems are fragments of modal predicate logics the modal operators in which are applied to closed formulas only. From a technical viewpoint fragments of modal predicate logics of this type can be reduced to the underlying fragment of predicate logic and the propositional modal logic part, see Finger and Gabbay (1992).

The systems investigated in Baader and Laux (1995) and Wolter and Zakharyaschev (1998a), (1998b), (1998c) allow the application of modal operators to both to axioms as well as concepts. The expressive power of the resulting language is basically the same as the one we obtain from **CPDLC** by omitting global role names and introducing the universal role name U: the interpretation $U^{\mathcal{M}(w)}$ of the universal role U is fixed for all models $\langle \mathcal{F}, \mathcal{M} \rangle$ and equals $\Delta \times \Delta$, for every state $w \in W$. The satisfaction problem of the resulting system is decidable, this can be proved by a straightforward modification of the proof in Wolter and Zakharyaschev (1998a). Observe that the omission of global role names is essential for the decidability of systems having the universal role: the system we obtain from **CPDLC** by keeping the global role names and adding the universal role has an undecidable satisfaction problem, since $\mathbf{K}_u \times \mathbf{K}_u$ is embedded in it. Now we can also explain the undecidability of the satisfaction problem of the expansion of **CPDLC** by roles of the form $[\alpha]R$. Let C be a concept of **CPDLC** expanded by the universal role. Assume that the action variables of C are contained in $\alpha_1, \ldots, \alpha_n$ and that R is a (local or global) role not contained in C. Put

$$D = [(\alpha_1 \cup \alpha_1^- \cup \ldots \cup \alpha_n^-)^*][\alpha_{n+1}]\bot$$

and denote by $D^\sharp$ the conjunction of D and $C^\sharp$, where $C^\sharp$ is the concept which results from C when all occurrences of U are replaced with $[\alpha_{n+1}]R$. Then C is satisfiable iff $D^\sharp$ is satisfiable and so the satisfaction problem for **CPDLC** expanded by roles of the form $[\alpha]R$ is undecidable. (Observe that $[\alpha_{n+1}]R$ behaves like the universal role for all relevant states.)

8 Open problems

(1) In this paper we have shown the decidability of $L \times \mathbf{K}_m$ for various rather expressive modal logics L. The complexity of the decision problems remains

open. The decision procedures delivered are easily seen to be non–elementary, and we conjecture that the decision problems for **CPLDC** and $\mathbf{S5}_n^C \times \mathbf{K}_m$ are in fact non–elementary. We have no specific conjecture for $\mathbf{S4} \times \mathbf{K}$, the decision problem of which is known to be NEXPTIME–hard, see Marx (1997). Also the complexity of the products of temporal logics and polymodal **K** considered in this paper are NEXPTIME–hard but matching upper bounds are not known.

(2) The system **CPDLC** is certainly not yet fine–tuned enough to serve as a logic of action. It would be of interest to investigate extensions of this logic by means of operations considered already in the propositional case, see e.g., Prendinger and Schurz (1996) and de Giacomo and Lenzerini (1995)

(3) The logic **CPDLC** is based on the minimal description logic containing all the booleans, namely $\mathcal{ALC}$. Often, however, more expressive power is required, e.g., number restrictions, converse roles, transitive roles, reflexive transitive closure of roles. It would be of interest to study the decision problem for converse **PDL** based on extensions of $\mathcal{ALC}$. Actually it is not difficult to extend our results to $\mathcal{ALC}$ extended by converse roles, and we conjecture that similar results hold for extensions by means of number restrictions. On the other hand, allowing reflexive transitive closure of roles leads to an undecidable system (again $\mathbf{K}_u \times \mathbf{K}_u$ is embeddable). A precise picture of the landscape for extensions of $\mathcal{ALC}$ remains to be drawn.

(4) It follows from the proof of Theorem 2.5 that **PDLK** is properly contained in the set $\mathbf{PDLK}_f$ of all formulas which are valid in products of the form $\mathcal{F} \times \mathcal{G}$ such that $\mathcal{G}$ is based on a finite domain Δ. Given that in many application domains the states are assumed to be finite, the logic $\mathbf{PDLK}_f$ is certainly of interest and it would be desirable to have a decision procedure for it. The same applies to the products $\mathbf{S5}_n^C \times \mathbf{K}_m$ and the products of temporal logics and polymodal **K**.

(5) In this paper we have considered the decision problem only. The finite model property remains open and (non–trivial) axiomatizations would be desirable. We believe that the technique developed so far is also helpful to attack those problems.

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