Fragments of common knowledge logics

Frank Wolter
Institut für Informatik, Universität Leipzig
Augustus-Platz 10-11, 04109 Leipzig, Germany
e-mail: wolter@informatik.uni-leipzig.de

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1 Introduction

Common knowledge logics are modal logics with one knowledge operator $\mathbf{K}_i$ for each agent (or player) i and common knowledge operators $\mathbf{C}_M$ for each set M of at least two agents. The intended meaning of $\mathbf{K}_i\varphi$ and $\mathbf{C}_M\varphi$ is "agent i knows (believes) φ" and "it is common knowledge among the agents in M that φ", respectively. Propositional common knowledge logics have been investigated intensively (cf. [6], [2], [14], [10], [11]). They turned out to be important for the epistemic analysis of games, the philosophy of knowledge, and the analysis of Multi-Agent systems in Computer Science and Artificial Intelligence.

On the other hand, first order common knowledge logics have been investigated only recently and so far only from the proof theoretic viewpoint, see [11] and [12]. In this paper we start a model theoretic investigation of (fragments of) first order common knowledge logics. First order extensions of propositional common knowledge logics are of interest both from the point of view of applications as well as from the viewpoint of pure logic: of course, we require first order extensions of propositional common knowledge logics whenever the application domain is infinite. An example is the epistemic analysis of the playability of games with mixed stategies, where quantification over the real numbers (or at least a real closed field) is required, see [12]. Also first order logic is required to represent knowledge (or common knowledge) about a finite domain the cardinality of which is not known in advance. This applies to most application domains which appear in Knowledge Representation, consider e.g., a domain consisting of the employees of a company, or the set of mails in a mailbox.

The following two questions on first order common knowledge logics are addressed: (1) we discuss the complexity the decision problem for various fragments of first order common knowledge logics and (2) we discuss some formal properties of "quantifying in". The first question is standard from the viewpoint of pure logic and the second question is related to the the study of "term existence theorems" in pure logic; that is to say, the question whether an existential quantifier can be replaced with a term. However, both problems are relevant to possible applications of first order common knowledge logics.

- From the viewpoint of its applicability in Knowledge Representation the decision problem is of course of extreme importance. Applicable systems require the design of an inference engine and so the logic should be decidable or at least recursively enumerable. But also from a more theoretical viewpoint—e.g., for the epistemic analysis of games—the decidability (or recursive enumerability) of a logic of knowledge is of interest: for example, if a common knowledge logic L is not recursively enumerable then (for all the cases considered here) also the set

  $$\{\psi : \mathbf{K}_1\psi \in L\}$$

  is not recursively enumerable. That is to say, the set of sentences an agent would know without making any assumptions about the world is not recursively enumerable. The agent would—in a sense—be considerably more intelligent than a Turing machine which might be a reason to reject L as a reasonable system of knowledge for certain applications. In general, results about the decision problem for a logic of knowledge show the implicit computational abilities of the agents one assumes by applying a certain logic of knowledge (or belief).

It will be shown that semantically defined full first order common knowledge logics are in fact extremely complex. In contrast to the (easily proved) fact that all the standard modal epistemic logics without common knowledge operators are finitely axiomatizable (and so recursively enumerable), the tautologies of all reasonable semantically defined first order common knowledge logics are not recursively enumerable. Given this high complexity of the full systems we look for expressive fragments which are less complex and so both more practicable as well as theorically justified. A similar question—i.e., the search for expressive decidable fragments of first order logic—has been in the center of interest in classical predicate logic since its undecidability was proved, see [3]. We shall follow this path here and study the following fragments of first order common knowledge logic:

- finite variable fragments,
- fragments with unary (alias monadic) predicates only.
- fragments with constant symbols only.
- fragments which allow to "quantify into" epistemic contexts with one variable only.

It will turn out that only the last option leads towards various decidable but still rather expressive and natural common knowledge logics.

- A language allows to "quantify into" epistemic contexts iff it allows to apply epistemic operators to open formulas. The possibility to "quantify into" epistemic contexts is the main reason for the expressive power of first order common knowledge logics. In fact, first order common knowledge languages which allow epistemic operators to be applied to closed formulas only are just simple combinations of propositional common knowledge logics and classical first order logic without any new interesting properties. To illustrate "quantifying in" we give an example which was worked out and analysed in detail in a proof theoretic manner by Kaneko and Nagashima, see [11] and [12]. Consider a finite game G with mixed strategies. Denote by Γ the theory of real closed fields and by $\mathsf{Nash}_G(\overline{x})$ a formula which says that the vector $\overline{x}$ is a Nash Equilibrium for G. We have $\Gamma \models \exists\overline{x}\ \mathsf{Nash}_G(\overline{x})$. For all the common knowledge logics we consider in this paper we can deduce $\mathbf{C}\Gamma \models \mathbf{C}\exists\overline{x}\ \mathsf{Nash}_G(\overline{x})$. (Here and in what follows we denote by $\mathbf{C}\Gamma$ the set $\{\mathbf{C}\psi : \psi \in \Gamma\}$.) This means that the existence of a Nash Equilibrium for G is common knowledge whenever the theory of real closed fields is common knowledge. This does not imply, however, that a Nash Equilibrium $\overline{x}$ for G is common knowledge. That is to say, we do not necessarily have

$$\mathbf{C}\Gamma \models \exists \overline{x}\ \mathbf{C}\mathsf{Nash}_G(\overline{x}). \tag{1}$$

Whether (1) holds depends on what is assumed about the common knowledge about the theory of real closed fields as well as the payoff functions of the agents.

Obviously there are numerous contexts which give rise to the question whether we have—for a set of formulas X and a non–epistemic formula $\varphi(x)$—

$$X \models \exists x \mathbf{C}\varphi(x)\ ? \tag{2}$$

This is the easiest and probably the most interesting form of "quantifying in". In this paper we start a semantic investigation of consequences of this type.

It should be observed that most of the results presented in this paper do not depend on the behaviour of the epistemic operators but on the non–epistemic vocabulary and the interaction between quantifiers and modal operators only. So it does not make any difference for our investigation whether we model belief or knowledge or whether we assume positive introspection, negative introspection, or both.

2 Syntax and Semantics of the full language

We define the syntax of first order common knowledge logic. The full language $\mathcal{CEL}^=$ is constructed from

- local predicate symbols $P_1, \ldots,$
- global predicate symbols $Q_1, \ldots,$
- local function symbols $f_1, \ldots,$
- global function symbols $g_1, \ldots,$
- infinitely many variables $x_1, \ldots,$
- the logical constants $\wedge$, $\Rightarrow$, $\neg$,
- the equality symbol $=$,
- a quantifier $\forall x$ for each variable x,
- knowledge operators $\mathbf{K}_1$ and $\mathbf{K}_2$,

- the common knowedge operator $\mathbf{C}$.

Observe that we assume two agents only. However, everything we prove in this paper for (fragments of) $\mathcal{CEL}^=$ is easily seen to hold for arbitrarily many agents as well. The predicate and function symbols come equipped with integers indicating their arity. Function symbols of arity 0 are called constants. *Terms* are defined inductively as follows:

- every variable and constant is a term.
- if f is a (local or global) function symbol of arity n and $t_1, \dots, t_n$ are terms, then $ft_1, \dots, t_n$ is a term.

A *global term* is a term which is composed from variables and global constants by means of global function symbols. Thus variables are global terms. Put $\mathbf{P}\varphi = \neg\mathbf{C}\neg\varphi$, for any formula φ.

We briefly explain the intuitive difference between global and local symbols. (Logics will be defined semantically in this paper, so the precise definition of the meaning of global and local symbols will be given later via their interpretation in models.) The extension of a global predicate symbol as well as the interpretation of a global function symbol is assumed to be common knowledge. That is to say,

$$\forall\overline{x}(\mathbf{P}Q(\overline{x}) \Rightarrow Q(\overline{x}) \wedge Q(\overline{x}) \Rightarrow \mathbf{C}Q(\overline{x})) \tag{3}$$

is valid, for all global predicate symbols Q, and

$$\forall\overline{x}\forall y(\mathbf{P}(g(\overline{x}) = y) \Rightarrow (g(\overline{x})) = y) \wedge (g(\overline{x}) = y) \Rightarrow \mathbf{C}(g(\overline{x}) = y)) \tag{4}$$

is valid, for all global function symbols g. The equality symbol is a special global predicate symbol, and so equality between objects is assumed to be common knowledge.

We now define the semantics for the language $\mathcal{CEL}^=$. A *state* S is a first order structure

$$S = \langle D, P_1^S \dots, Q_1^S, \dots, f_1^S \dots, g_1^S, \dots\rangle,$$

such that P_i^S as well as Q_i^S are subsets of D^n (n the arity of P_i and Q_i) and f_i^S as well as g_i^S are functions from D^n into D (n the arity of f_i and g_i).

A propositional Kripke structure is a relational structure

$$\mathcal{F} = \langle W, R_1, R_2\rangle,$$

where W is a non-empty set of worlds and R_1, R_2 are binary relations between worlds. Propositional common knowledge logic (i.e., propositional logic extended by means of the operators $\mathbf{K}_1$, $\mathbf{K}_2$, and $\mathbf{C}$) is interpreted in propositional Kripke structures in the standard manner, see e.g. [6]. The intended meaning of $w_1 R_i w_2$ is: in world w_1 agent i believes (or knows) that w_2 is possible. Given a class of propositional Kripke frames $\mathcal{C}$ we denote by $L_p(\mathcal{C})$ the propositional common knowledge determined by $\mathcal{C}$. Various logics of the form $L_p(\mathcal{C})$ have been investigated intensively: by **K.C** we denote the propositional logic determined by the class of all propositional Kripke frames and by **K4.C**, **S4.C**, **K4D.C**, **K45D.C**, and **S5.D** we denote the propositional logic determined by the class of

- transitive,
- transitive and reflexive,
- transitive and serial,
- transitive, euclidean and serial,
- symmetric, transitive, and reflexive structures,

respectively. All these logics have been axiomatized and are known to be decidable, see [6].

We are going to combine first order structures and propositional Kripke structures in order to obtain models for first order common knowledge logic. This is done in a straightforward manner by associating with any world in a propositional Kripke structure a state representing the structure of that world: A *first order Kripke structure* $\langle \mathcal{F}, D, S \rangle$ consists of a propositional Kripke structure $\mathcal{F}$, a non-empty set of objects D and a mapping S associating with each $w \in W$ a state

$$S(w) = \langle D, P_1^{S(w)} \ldots, Q_1^{S(w)}, \ldots, f_1^{S(w)} \ldots, g_1^{S(w)}, \ldots \rangle,$$

such that $Q_i^{S(w)} = Q_i^{S(v)}$ and $g_i^{S(w)} = g_i^{S(v)}$, for all $w, v \in W$ and $i \in \omega$. This reflects the fact that the interpetation of global symbols does not vary between worlds. In the situation above we say that $\langle \mathcal{F}, D, S \rangle$ is based on $\mathcal{F}$. Observe that we assume constant domains; that is to say, the domain D is not allowed to vary from state to state. This assumption means that the domain is common knowledge and is syntactically reflected by the validity of the Barcan formula:

$$\forall \overline{x} \mathbf{C} \varphi \Leftrightarrow \mathbf{C} \forall \overline{x} \varphi.$$

A valuation $\mathcal{V}$ into $\langle \mathcal{F}, D, S \rangle$ is a mapping from the set of variables into D. Define by induction the interpretation $t^{S(w)}$ of a term t in a world w and the relation $\models_{\mathcal{V}}$ between worlds and formulas:

- for all variables x: $x^{S(w)} = \mathcal{V}(x)$.
- for all function symbol f of arity n and terms $t_1, \ldots, t_n$,

$$f(t_1, \ldots, t_n)^{S(w)} = f^{S(w)}(t_1^{S(w)}, \ldots, t_n^{S(w)}).$$

- $w \models_{\mathcal{V}} t_1 = t_2$ iff $t_1^{S(w)} = t_2^{S(w)}$, for all terms t_1, t_2.

- $w \models_{\mathcal{V}} P(t_1, \ldots, t_n)$ iff $\langle t_1^{S(w)}, \ldots, t_n^{S(w)} \rangle \in P^{S(w)}$.
- the steps for conjunction, negation, implication, and quantification are as usual.
- $w \models_{\mathcal{V}} \mathbf{K}_i\varphi$ iff $v \models \varphi$, for all v with wR_iv, for $i \in \{1, 2\}$.
- $w \models_{\mathcal{V}} \mathbf{C}\varphi$ iff $v \models \varphi$, for all v with $w(R_1 \cup R_2)^*v$. $(R_1 \cup R_2)^*$ is the reflexive and transitive closure of $R_1 \cup R_2$.

A closed formula φ is valid in a structure $\mathcal{G}$, in symbols $\mathcal{G} \models \varphi$, if $w \models \varphi$, for every world w in $\mathcal{G}$.

For a class of propositional Kripke structures $\mathcal{C}$ we denote by $L(\mathcal{C})$ the set of closed formulas valid in all first order Kripke structures based on structures in $\mathcal{C}$. Given a propositional logic $L = L_p(\mathcal{C})$—where $\mathcal{C}$ is the class of all propositional Kripke structures validating L—we put $L^Q = L(\mathcal{C})$. So $\mathbf{K}.\mathbf{C}^Q$ is the logic determined by the class of all first order Kripke structures. Let $L = L(\mathcal{C})$, Γ a set of formulas, and φ a formula. Then φ follows from Γ in L, in symbols $\Gamma \models_L \varphi$, iff for all $\mathcal{F} \in \mathcal{C}$, all first order Kripke structures $\langle \mathcal{F}, D, S \rangle$ based on $\mathcal{F}$, all valuations $\mathcal{V}$, and all worlds w:

$$w \models_{\mathcal{V}} \Gamma \text{ implies } w \models_{\mathcal{V}} \varphi.$$

We now define the notion of a (standard) first order common knowledge logic. The definition was chosen for technical reasons and we do not claim that it covers all reasonable first order common knowledge logics.

DEFINITION 2.1 (First order common knowledge logic) A set $L \subseteq \mathcal{CEL}^=$ is called a first order common knowledge logic iff there exists a class $\mathcal{C}$ of propositional Kripke frames containing all frames of the form $\langle W, R_1, R_2 \rangle$, R_1 and R_2 equivalence relations, such that $L = L(\mathcal{C})$.

The class $\mathcal{STD}$ of standard first order common knowledge logics is defined by putting

$$\mathcal{STD} = \{\mathbf{K}.\mathbf{C}^Q,\ \mathbf{K4}.\mathbf{C}^Q,\ \mathbf{S4}.\mathbf{C}^Q,\ \mathbf{K4D}.\mathbf{C}^Q,\ \mathbf{K45D}.\mathbf{C}^Q,\ \mathbf{S5}.\mathbf{D}^Q\ \}.$$

All the results we are going to formulate will either hold for the class of all first order common knowledge logics or for the class of all standard ones. Hence they do not depend on the specific properties of the knowledge operators.

3 An example

We formalize a 2–person finite strategic–form game G with mixed strategies. Recall that such a game consists of two finite sets $S_1 = \{s_1^1, \ldots, s_{l(1)}^1\}$ and $S_2 = \{s_2^2, \ldots, s_{l(2)}^2\}$ of strategies and payoff functions u_i, $i = 1, 2$, from $S = S_1 \times S_2$

into the set of real numbers. $u_i(s_1, s_2)$ is the payoff of player i from the pure strategy combination $\langle s_1, s_2 \rangle \in S_1 \times S_2$.

G can be represented on the basis of the language $\mathcal{L}_R$ of real closed fields. Recall that this consists of

- function symbols: 0^r, 1^r, $+^r$, $-^r$, $\cdot^r$, $/^r$,
- predicate symbols: $\leq^r$.

The intended meaning of those symbols should be clear. We shall mostly omit the superscript r. The theory of real closed fields is complete. Hence it coincides with the theory of the real numbers. Observe that we do not fix yet whether the symbols are global or local. Of course, in classical logic this distinction is meaningless. To formalize G we require a new set of *constant symbols*

- $[u_i(s_1, s_2)]$, for $s_1, s_2 \in S_1 \times S_2$ and $i \in \{1, 2\}$

which are used to denote the payoff for player i from the pure strategy combination $\langle s_1, s_2 \rangle$. A *mixed strategy* of player i is a vector of terms $\bar{t}_i = \langle t_{i1}, \ldots, t_{il(i)} \rangle$ such that

$$\mathsf{St}(\bar{t}_i) = (\sum_{k=1}^{l(i)} t_{ik} = 1) \wedge (\bigwedge \{t_{ik} \geq 0 : 1 \leq k \leq l(i)\}).$$

The *expected payoff* to player i from a mixed strategy combination $\bar{t} = \langle \bar{t}_1, \bar{t}_2 \rangle$ is given by the term

$$G_i(\bar{t}) = \sum_{k_1} \sum_{k_2} t_{1k_1} \cdot t_{2k_2} \cdot [u_i(s^1_{k_1}, s^2_{k_2})].$$

Now we have all the basic ingredients required to formalize G. For example, given the mixed strategy combination $\bar{t} = \langle \bar{t}_1, \bar{t}_2 \rangle$ we define

$$\begin{aligned} \mathsf{Nash}_G(\bar{t}) \quad = \quad & \mathsf{St}(\bar{t}_1) \wedge \forall \bar{x}_1 (\mathsf{St}(\bar{x}_1) \Rightarrow (G_1(\bar{t}) \geq G_1(\bar{x}_1, \bar{t}_2)) \\ & \wedge \mathsf{St}(\bar{t}_2) \wedge \forall \bar{x}_2 (\mathsf{St}(\bar{x}_2) \Rightarrow G_2(\bar{t}) \geq G_2(\bar{x}_2, \bar{t}_1)). \end{aligned}$$

This formula says that $\bar{t}$ is a Nash Equilibrium for G. Denote by Φ_r an axiomatization of the theory of real closed fields (with the equality symbol $=$). Then

$$\Phi_r \models \exists \bar{x} \mathsf{Nash}_G(\bar{x})$$

holds for every game G. This follows from the facts that every game has a Nash Equilibrium in the real numbers and that the theory of real closed fields is complete.

So far we did not use the epistemic part of the language. This is required, however, to give an epistemic analysis of the game. In what follows we assume that an axiomatization Φ_r of the theory of real closed fields is common knowledge and ask for various sentences φ whether they follow (in a logic L) from $\mathbf{C}\Phi_r$ or not. Of course, we have for any finite game G:

$$\mathbf{C}\Phi_r \models_L \mathbf{C}\exists\overline{x}\mathsf{Nash}_G(\overline{x});$$

that is to say, it is common knowledge that for every game there exists a Nash Equilibrium. When does it follow that there exists a vector $\overline{x}$ such that it is common knowledge that $\overline{x}$ is a Nash Equilibrium for G?

It turns out that this depends on whether the function and predicate symbols involved are global or local. Given a propositional frame $\langle W, R_1, R_2\rangle$ and $w \in W$ we say that $v \in W$ is *reachable* from w if $w(R_1 \cup R_2)^*v$. Now we have the following cases:

- Suppose all function and predicate symbols in Φ_r and $\mathsf{Nash}_G(\overline{x})$ are local symbols. Then $w \models \mathbf{C}\Phi_r$ means that all the worlds v reachable from w are real closed fields with the same domain, but that the interpretation of the symbols may vary between the worlds. In this case $\mathbf{C}\Phi_r \models_L \exists\overline{x}\mathbf{C}\mathsf{Nash}_g(\overline{x})$ holds iff $\Phi_r \models \forall\overline{x}\mathsf{Nash}_G(\overline{x})$ holds in classical logic. (This follows from Theorem 4.1 below.) Thus, under the local interpretation of function and predicate symbols there are no interesting games the Nash Equilibria of which are common knowledge. One explanation seems to be that the constants $[u_i(s_1, s_2)]$ are local. Hence the players do not know the payoff functions of the game and so we should not expect them to know the Nash Equilibria for the game. However, it is not difficult to extend the technique of the proof of Theorem 4.1 to show that the same result holds under the assumption that the constants $u_i[s_1, s_2]$ are global.

- Suppose all function symbols in Φ_r and $\mathsf{Nash}_G(\overline{x})$ are interpreted globally. Then $\leq^r$ is also interpreted globally, as is readily checked. Then $w \models \mathbf{C}\Phi_r$ means that all the worlds v reachable from w actually coincide and obviously $\mathbf{C}\Phi_r \models_L \exists\overline{x}\mathbf{C}\mathsf{Nash}_G(\overline{x})$ holds for all L. In this case "quantifying into" is trivial.

- Again we assume that all function symbols in Φ_r and $\mathsf{Nash}_G(\overline{x})$ are interpreted globally. However, this time we do not use the equality symbol to express equality, but we use a new (local) predicate symbol eq to express equality. Denote by φ^{eq} the formula which results from φ when all occurences of the equality symbol in φ are replaced with eq. Denote by $A(\mathsf{eq})$ a set of formulas axiomatizing equality in first order logic. Then

$$\mathbf{C}(\Phi_r^{\mathsf{eq}} \cup A(\mathsf{eq})) \models_L \exists\overline{x}\mathbf{C}\mathsf{Nash}_G^{\mathsf{eq}}(\overline{x})$$

holds iff there exists a vector of closed terms $\overline{t}$ such that

$$\mathbf{C}(\Phi_r^{\mathsf{eq}} \cup A(\mathsf{eq})) \models_L \mathbf{C}\mathsf{Nash}_G(\overline{t}),$$

see Theorem 4.2 below. This case was considered in a proof theoretic manner in [12].

Thus various interesting properties of the resulting system depend on whether local or global symbols are chosen. If symbols of mathematical theories are assumed to be local, then most of the resulting properties of "quantifying in" can be explained by the fact that mathematics is not common knowledge. Thus, in order to model a situation in which mathematics is common knowledge, all the symbols of the background mathematical theory should be global. We illustrate the situation with global symbols in the mathematical theory by means of the following example.

Assume that all the symbols of Φ_r and $\mathsf{Nash}_G(\overline{x})$ are global. Introduce new vectors of local constants $\overline{a}_i = \langle a_{i1}, \ldots, a_{il(i)} \rangle$, for $i = 1, 2$. $\overline{a}_i$ is intended to denote the mixed strategy chosen by player i. To express that player i knows his own mixed strategy the following abbreviation is required.

Let Y_1 and Y_2 be sets of local predicate and function symbols, respectively. Let, for $i \in \{1, 2\}$, $\mathsf{Glob}_i(Y_1, Y_2)$ be the union of the following two sets:

$$\{\forall \overline{x}(\neg \mathbf{K}_i \neg P(\overline{x}) \Rightarrow P(\overline{x}) \wedge P(\overline{x}) \Rightarrow \mathbf{K}_i P(\overline{x}) : P \in Y_1\},$$

$$\{\forall \overline{x} \forall y(\neg \mathbf{K}_i \neg f(\overline{x}) = y \Rightarrow f(\overline{x})) = y \wedge f(\overline{x}) = y \Rightarrow \mathbf{K}_i f(\overline{x}) = y : f \in Y_2\}.$$

It should be clear that $\mathsf{Glob}_i(Y_1, Y_2)$ says that the extensions of the symbols in Y_1, Y_2 are known to player i. Denote by Γ the following set:

- Φ_r, $\mathsf{Glob}_1(\emptyset, \{\overline{a}_1\})$, $\mathsf{Glob}_2(\emptyset, \{\overline{a}_2\})$,
- $\forall \overline{y}_1(\mathsf{St}(\overline{y}_1) \Rightarrow (G_1(\overline{a}) \geq G_1(\overline{y}_1, \overline{a}_2)), \forall \overline{y}_2(\mathsf{St}(\overline{y}_2) \Rightarrow (G_2(\overline{a}) \geq G_2(\overline{a}_1, \overline{y}_2))$.
- $\mathsf{St}(\overline{a}_1)$, $\mathsf{St}(\overline{a}_2)$.

Of course we have $\mathbf{C}\Gamma \models_L \mathbf{C}\mathsf{Nash}_G(\overline{a})$ and $\mathbf{C}\Gamma \models_L \exists \overline{x} \mathsf{Nash}_G(\overline{x})$. But we do not have $\mathbf{C}\Gamma \models_L \exists \overline{x} \mathbf{C} \overline{x}(\overline{x} = \overline{a})$, i.e., the final decision $\overline{a} = \langle \overline{a}_1, \overline{a}_2 \rangle$ is not common knowledge. A counterexample can be constructed using a game G with two Nash equilibria $\langle \overline{a}_1, \overline{a}_2 \rangle$ and $\langle \overline{a}_1, \overline{a}'_2 \rangle$ with $\overline{a}_2 \neq \overline{a}'_2$.

4 Quantifying into epistemic contexts

In this section we prove various (non–epistemic) conditions equivalent to the implication $\Gamma \models_L \exists x \mathbf{C} \varphi(x)$. First some notation is required. A formula (a set of formulas) is called non–epistemic if it does not contain epistemic operators. A formula is *without quantifying in* if it does not contain a subformula of the form $\mathbf{K}_i \psi$ or $\mathbf{C} \psi$ such that ψ is open.

In what follows we assume that Γ is a set of closed formulas and $\varphi(x)$ is a formula containing the free variable x only. The following theorem states that no non-trivial assertion of the form $\exists x \mathbf{C} \varphi(x)$ follows from a theory without "quantifying in" and global symbols. Moreover, even the addition of a weak form of "quantifying in" does not change this situation. Namely, we allow that certain symbols are global relative to one of the players.

THEOREM 4.1 *(I) Let L be a first order common knowledge logic and suppose φ and Γ are non-epistemic and do not contain global function symbols nor global predicate symbols (but possibly equality). Suppose that Y_1^1, Y_1^2 are disjoint sets of local predicate symbols which do not occur in Γ and that Y_2^1, Y_2^2 are disjoint sets of local function symbols which do not occur in Γ. Then the following conditions are equivalent:*

- $\mathbf{C}\Gamma \models_L \exists x \mathbf{C}\varphi(x)$.
- $\mathbf{C}(\Gamma \cup \mathsf{Glob}_1(Y_1^1, Y_2^1) \cup \mathsf{Glob}_2(Y_1^2, Y_2^2)) \models_L \exists x \mathbf{C}\varphi(x)$.
- $\Gamma \models \forall x \varphi(x)$ *holds in classical first order logic.*
- $\mathbf{C}\Gamma \models_L \forall x \mathbf{C}\varphi(x)$.
- *For all terms t:* $\mathbf{C}\Gamma \models_L \mathbf{C}\varphi(t)$.

(II) Suppose $L \in \mathcal{STD}$, φ is non-epistemic, Γ is closed and without quantifying in and both do not contain global function symbols nor global predicate symbols. Then the following conditions are equivalent:

- $\Gamma \models_L \exists x \mathbf{C}\varphi(x)$.
- $\Gamma \models_L \forall x \mathbf{C}\varphi(x)$.
- *For all closed terms t:* $\mathbf{C}\Gamma \models_L \mathbf{C}\varphi(t)$.

Proof In this proof global symbols (save the equality symbol) are not required. So it suffices to take first order structures which interpret local symbols only. We require the following definition. Suppose

$$S = \langle D, P_1^S, \ldots, f_1^S \ldots \rangle$$

is a first order structure and $a \in D$. Then we define, for $b \in D$, a first order structure S_{ab} as follows. Let $\sigma : D \to D$ be the mapping defined by putting $\sigma(d) = d$, for all $d \in D - \{a, b\}$, $\sigma(a) = b$, $\sigma(b) = a$. Define $P_i^{S_{ab}}$ and $f_i^{S_{ab}}$, $i \in \omega$, by putting for all $d_1, \ldots, d_k, d \in D$ and all P_i and f_i of arity k,

$$P_i^{S_{ab}}(\sigma(d_1), \ldots, \sigma(d_k)) \text{ iff } P_i^S(d_1, \ldots, d_k),$$

and

$$f_i^{S_{ab}}(\sigma(d_1), \ldots, \sigma(d_k)) = \sigma(d) \text{ iff } f_i^S(d_1, \ldots, d_k) = d.$$

Now let

$$S_{ab} = \langle D, P_1^{S_{ab}} \ldots, f_1^{S_{ab}} \ldots \rangle.$$

We prove (I). (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (5), and (5) $\Rightarrow$ (1) are obvious. It remains to show (2) $\Rightarrow$ (3). Assume that $\forall x\varphi(x)$ does not follow (in classical logic) from Γ. We are going to prove that (2) does not hold. Assume for simplicity that $Y_1^1 = \{h_1\}$, $Y_2^1 = \{L_1\}$, $Y_1^2 = \{h_2\}$, and $Y_2^2 = \{L_2\}$. By $P_1, \ldots$ and $f_1, \ldots$ we denote the remaining local symbols.

Take a first order structure

$$S = \langle D, P_1^S, \ldots, L_1^S, L_2^S, f_1^S \ldots, h_1^S, h_2^S \rangle$$

such that $S \models \Gamma$ but $S \not\models \forall x\varphi(x)$. Take $a \in D$ such that, for a valuation $\mathcal{V}$ with $\mathcal{V}(x) = a$, we have $S \models_{\mathcal{V}} \neg\varphi(x)$. We may assume (by Löwenheim–Skolem) that $D = \{b(0), b(1), \ldots\}$.

For each $b \in D$ define a valuation $\mathcal{V}_b$ in S_{ab} by putting $\mathcal{V}_b(x) = b$. Obviously

$$S_{ab} \models_{\mathcal{V}_b} \neg\varphi(x).$$

Define $\langle W, R_1, R_2 \rangle, D, S' \rangle$ by putting

- $W = \{0, 1, \ldots\})$,
- $R_1 = \bigcup\{\{2n, 2n+1\} \times \{2n, 2n+1\} : n \in \omega\}$,
- $R_2 = \bigcup\{\{2n+1, 2n+2\} \times \{2n+1, 2n+2\} : n \in \omega\}$,
- $S'(2n) = S_{ab(n)}$, for all n,
- $S'(2n{+}1) = \langle D, P_1^{S_{ab(1)}}, \ldots, L_1^{S_{ab(n)}}, L_2^{S_{ab(n+1)}}, f_1^{S_{ab(n)}}, \ldots, h_1^{S_{ab(n)}}, h_2^{S_{ab(n+1)}} \rangle$, for all n.

It remains to observe $0 \models \mathbf{C}(\Gamma \cup \mathsf{Glob}_1(Y_1^1, Y_2^1) \cup \mathsf{Glob}_2(Y_1^2, Y_2^2))$ and $0 \not\models \exists x\mathbf{C}\varphi(x)$.

Proof of (II). (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are obvious. It remains to show (1) $\Rightarrow$ (2). Suppose we do not have $\Gamma \models_L \forall x\mathbf{C}\varphi(x)$. Take a first order Kripke structure $\langle\langle W, R_1, R_2 \rangle, D, S\rangle$ such that $w \models \Gamma$ but $w \not\models \forall x\mathbf{C}\varphi(x)$. Take a v which is reachable from w such that $v \models_{\mathcal{V}} \neg\varphi(x)$, where $\mathcal{V}(x) = a$. We define a first order Kripke structure $\langle\langle W', R_1', R_2' \rangle, D, S'\rangle$ by putting $W' = W \cup D$,

$$w_1 R_i' w_2 \Leftrightarrow \begin{cases} w_1 R_i w_2 & : \quad w_1, w_2 \in W \\ v R_i w_2 & : \quad w_1 \in D, w_2 \in W \\ w_1 R_i v & : \quad w_2 \in D, w_1 \in D \\ v R_i v & : \quad w_1, w_2 \in D \end{cases}$$

$$S'(w) = \begin{cases} S(w) & : \quad w \in W \\ S_{ab} & : \quad w = b \in D \end{cases}$$

It is not difficult to check that $\langle\langle W', R_1', R_2' \rangle, D, S'\rangle$ validates L whenever $L \in \mathcal{STD}$. Moreover, $w \models' \Gamma$ but not $w \models' \exists x\mathbf{C}\varphi(x)$, where $\models'$ is the satisfaction relation in the new model. ❏

In the following theorem we allow global function symbols but no equality. In this case non-trivial assertions of the form $\exists x \mathbf{C}\varphi(x)$ follow from theories without (explicit) "quantifying in"; namely $\exists x \mathbf{C}\varphi(x)$ follows from a theory iff there is a global term t such that $\varphi(t)$ follows from that theory. A similar result was obtained in a proof theoretic manner in [11].

THEOREM 4.2 *Let L be a first order common knowledge logic and suppose that φ and Γ are non-epistemic and do not contain equality nor global predicate symbols. Suppose that Y_1^1, Y_1^2 are disjoint sets of local predicate symbols which do not occur in Γ and that Y_2^1, Y_2^2 are disjoint sets of local function symbols which do not occur in Γ. Then the following conditions are equivalent:*

- $\mathbf{C}\Gamma \models_L \exists x \mathbf{C}\varphi(x)$.
- $\mathbf{C}(\Gamma \cup \mathsf{Glob}_1(Y_1^1, Y_2^1) \cup \mathsf{Glob}_2(Y_1^2, Y_2^2)) \models_L \exists x \mathbf{C}\varphi(x)$.
- *there exists a global term t such that $\Gamma \models \varphi(t)$ hold in classical logic.*
- *there exists a global term t such that $\mathbf{C}\Gamma \models_L \mathbf{C}\varphi(t)$.*

Proof (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4), and (4) $\Rightarrow$ (1) are obvious. We show (2) $\Rightarrow$ (3). Assume for simplicity that $Y_1^1 = \{h_1\}$, $Y_2^1 = \{L_1\}$, $Y_1^2 = \{h_2\}$, and $Y_2^2 = \{L_2\}$. By $P_1, \ldots$ and $f_1, \ldots$ we denote the remaining local symbols and by $g_1, \ldots$ the global function symbols.

Suppose there does not exist a closed global term t such that $\Gamma \models_L \varphi(t)$. We find for each closed global term t a first order structure S_t with domain

$$D = \{s : \ s \text{ is a closed global term } \}$$

such that

$$g_i^{S_t}(t_1, \ldots, t_k) = g_i(t_1, \ldots, t_k),$$

for all global function symbols g_i of arity k and all closed terms $t_1, \ldots, t_k$ and $S_t \models \Gamma$ but $S_t \not\models \varphi(t)$. (Recall that such a term model exists since the language does not contain the equality symbol.) Now the construction of a model refuting (2) is similar to the previous proof. Let $t(0), t(1), \ldots$ be an enumeration of the closed global terms.

Define $\langle W, R_1, R_2 \rangle, D, S' \rangle$ by putting

- $W = \{0, 1, \ldots\}$,
- $R_1 = \bigcup \{\{2n, 2n+1\} \times \{2n, 2n+1\} : n \in \{0, 1, \ldots\}\}$,
- $R_2 = \bigcup \{\{2n+1, 2n+2\} \times \{2n+1, 2n+2\} : n \in \{0, 1, \ldots\}\}$,
- $S'(2n) = S_{t(n)}$, for all n,

- $S'(2n+1) =$
  $\langle D, P_1^{S_{t(n)}}, \ldots, L_1^{S_{t(n)}}, L_2^{S_{t(n+1)}}, f_1^{S_{t(n)}}, \ldots, h_1^{S_{t(n)}}, h_2^{S_{t(n+1)}}, g_1^{S_{t(n)}}, \ldots \rangle$, for all n.

Then $\langle\langle W, R_1, R_2\rangle, D, S'\rangle$ refutes (2). ❑

The following theorem shows that the two results presented above are optimal. We leave the construction of the counterexamples to the reader.

THEOREM 4.3 *Let L be an arbitrary common knowledge logic. Then the following holds.*

- *Let g be a unary global function symbol and $\Gamma = \{\exists x \forall y (g(y) \neq x)\}$, $\varphi = \forall y(g(y) \neq x)$. Then $\mathbf{C}\Gamma \models_L \exists x \mathbf{C}\varphi$ but there does not exist a closed global term t such that $\mathbf{C}\Gamma \models_L \mathbf{C}\varphi(t)$.*
- *Let P be a unary global predicate symbol and $\Gamma = \{\exists x P(x)\}$, $\varphi = P(x)$. Then $\mathbf{C}\Gamma \models_L \exists x \mathbf{C}\varphi$ but there does not exist a closed global term t such that $\mathbf{C}\Gamma \models_L \mathbf{C}\varphi(t)$.*
- *Let a_1 and a_2 be global constant symbols and P a local unary predicate symbol, $\Gamma = \mathbf{C}P(a_1) \vee \mathbf{C}P(a_2)$, $\varphi = P(x)$. Then $\Gamma \models_L \exists x \mathbf{C}\varphi$ but there does not exist a closed global term t such that $\Gamma \models_L \mathbf{C}\varphi(t)$.*

5 Finite variable fragments

The n–variable fragment of the language of first order (common knowledge) logic consists of all formulas containing n variables only. Finite variable fragments of classical (non-epistemic) logics have been investigated intensively. It turned out that the two variable fragment of classical first order logic is decidable; its three variable fragment, however, is undecidable, see [15], [9], and [3]. For common knowledge logics the situation is different; already the two variable fragments turn out to be undecidable. First the positive result:

THEOREM 5.1 *The one variable fragment without equality and function symbols of all logics in $\mathcal{STD}$ is decidable.*

This Theorem follows from the results in section 8.

The negative Theorem follows immediately from the results of Gabbay and Shehtman in [7] on the undecidability of two variable fragments of first order modal logics.

THEOREM 5.2 *The two variable fragment without equality and and function symbols of any first order common knowledge logics is undecidable.*

Observe that this result holds even for the fragments without the common knowledge operator.

6 Non-axiomatizability of the monadic fragment

The monadic fragment of a first order common knowledge logic L consists of all sentences in L containing only the equality symbol and unary local predicate symbols. It is well known that the monadic fragment of first order predicate logic is decidable. This can be proved by establishing its finite model property.

THEOREM 6.1 *The monadic fragment of any first order common knowledge logic L is not recursively enumerable.*

Proof Let L be an arbitrary first order common knowledge logic. We prove the result by embedding arithmetic into the monadic fragment of L. Firstly define a binary predicate symbol $<$ by putting

$$x < y := \mathbf{P}(P_1(x) \wedge P_2(y)).$$

Denote by φ_1 the conjunction of the following set of formulas which say that $<$ is a discrete strict linear ordering with a starting point but without endpoints:

- $\exists x \forall y (x < y \vee x = y)$, $\forall x (\neg (x < x)$,
- $\forall x \forall y \forall z (x < y \wedge y < z \Rightarrow x < z)$,
- $\forall x \forall y (x < y \vee y < x \vee x = y)$,
- $\forall x \exists y (x < y \wedge \neg \exists z (x < z \wedge z < y))$.

Put

$$\varphi_2 = \forall x \forall y (x < y \Leftrightarrow \mathbf{C}(x < y) \Leftrightarrow \mathbf{P}(x < y)).$$

φ_2 says that $<$ is global. Let E be a unary predicate symbol and let φ_3 denote the conjunction of the following set of formulas:

- $\psi_0 = \mathbf{C}(\forall x \forall y (E(x) \wedge y < x \Rightarrow E(y)))$.
- $\psi_1 = \mathbf{C}(\forall x (\neg \mathbf{K}_1 \neg E(x) \Rightarrow \forall y (y < x \Rightarrow E(y)))$.
- $\psi_2 = \forall x \mathbf{P} E(x)$.
- $\psi_3 = \forall x \neg E(x)$.
- $\psi_4 = \mathbf{C}(\forall x (\neg \mathbf{K}_2 \neg E(x) \Rightarrow \forall y (y < x \Rightarrow E(y)))$.

Finally define formulas

- $\mathsf{zero}(x) := \forall y \neg (y < x)$,
- $\mathsf{succ}(x, y) := x < y \wedge \neg \exists z (x < z \wedge z < y)$,
- $\mathsf{plus}(x, y, z) := \mathbf{P}(P_{11}(x) \wedge P_{21}(y) \wedge P_{31}(z))$.

- $\mathsf{mult}(x,y,z) := \mathbf{P}(P_{12}(x) \wedge P_{22}(y) \wedge P_{32}(z))$.

Let Γ be a set of sentences axiomating Peano Arithmetic using the predicates $<$, zero (which replaces the constant 0), succ (which replaces the successor function), plus (which replaces $+$), and mult (which replaces multiplication). Put $\Sigma = \Gamma \cup \{\varphi_1, \varphi_2, \varphi_3\}$. We show:

1. If $\langle\langle W, R_1, R_2\rangle, S, D\rangle$ is a model validating Σ in a world $w \in W$, then

$$\langle D, <^{S(w)}, \mathsf{plus}^{S(w)}, \mathsf{mult}^{S(w)}\rangle \simeq \langle \mathbf{N}, <, +, \cdot\rangle,$$

   where $\mathbf{N}$ is the set of natural numbers.

2. There exists a first order Kripke structure $\langle\langle W, R_1, R_2\rangle, S, D\rangle$ such that both R_1 and R_2 are equivalence relations and $w \in W$ with $w \models \Sigma$.

(1.) Let $\langle\langle W, R_1, R_2\rangle, S, D\rangle$ be a model validating Σ in w. We may assume assume that all $v \in W$ are reachable from w. Firstly we show that $\langle D, <^{S(v)}\rangle$ is isomorphic with $\langle \mathbf{N}, <\rangle$, for all $v \in W$.

Using φ_1 it is clear that $<^{S(w)}$ is a discrete strict linear ordering with a starting point and no endpoints. By φ_2, $<^{S(v)} = <^{S(v')}$, for all $v, v' \in W$.

- By ψ_3, $E^{S(w)} = \emptyset$,
- by ψ_0, $E^{S(v)}$ is downward closed (i.e., $b \in E^{S(v)}$ whenever $a \in E^{S(v)}$ and $b < a$), for all v.
- By ψ_2, for all $a \in D$ there exists a world v reachable from w such that $a \in E^{S(v)}$.

Hence it suffices to show that $E^{S(v)}$ is finite, for all v which are reachable from w. Suppose v is given. There exists a path $v_0 S_0 v_1 S_1 \ldots S_n v_{n+1}$ such that $v_0 = w$, $v_{n+1} = v$ and $S_i \in \{R_1, R_2\}$. By $\psi_1 \wedge \psi_4$ we have, for all $a, b \in D$,

$$a \in E^{S(v_{i+1})} \text{ and } b < a \text{ implies } b \in E^{S(v_i)}.$$

Hence the cardinality of $E^{S(v)}$ is bounded by n.

(2.) Define the structure $\langle\langle W, R_1, R_2\rangle, D, S\rangle$ by putting

- $W = D = \{0, 1, \ldots\}$,
- $R_1 = \bigcup\{\{2n, 2n+1\} \times \{2n, 2n+1\} : n \in \omega\}$,
- $R_2 = \bigcup\{\{2n+1, 2n+2\} \times \{2n+1, 2n+2\} : n \in \omega\}$,
- $E^{S(n)} = \{0, 1, \ldots n-1\}$,
- $P_1^{S(n)} = \{n\}$, $P_2^{S(n)} = \{n+1, \ldots\}$,

- Let $f = \langle f_1, f_2 \rangle$ be a bijection from $\mathbf{N}$ onto $\mathbf{N} \times \mathbf{N}$.

$$P_{11}^{S(n)} = \{f_1(n)\},\ P_{21}^{S(n)} = \{f_2(n)\},\ P_{31}^{S(n)} = \{f_1(n) + f_2(n)\},$$

$$P_{12}^{S(n)} = \{f_1(n)\},\ P_{22}^{S(n)} = \{f_2(n)\},\ P_{32}^{S(n)} = \{f_1(n) \cdot f_2(n)\}.$$

Then $0 \models \Sigma$. ❑

7 Non-axiomatizability of the fragment with constants only

The *fragment of constants* of a first order common knowledge logic L consists of all sentences in L which contain local constants and the equality symbol only.

THEOREM 7.1 *The fragment of constants of any first order common knowledge logic L is not recursively enumerable.*

Proof The proof is similar to the proof of Theorem 6.1. We define predicates $<$, E, mult, and plus and then consider the set Σ as defined there. Let

- $x < y := \mathbf{P}(x = c_1 \wedge y = c_2)$,
- $E(x) := \neg \mathbf{K}_1 \neg (x = c)$,
- $\mathsf{plus}(x, y, z) := \mathbf{P}(x = c_{11} \wedge y = c_{21} \wedge z = c_{31})$,
- $\mathsf{mult}(x, y, z) := \mathbf{P}(x = c_{12} \wedge y = c_{22} \wedge z = c_{32})$.

Define Σ as in the proof of Theorem 6.1. It remains to show (1.) and (2.) of the proof of Theorem 6.1. (1.) is proved in the same manner as above and so we consider (2.) only. Define the structure $\langle \langle W, R_1, R_2 \rangle, D, S \rangle$ by putting:

- $\Delta = \{0, 1, \ldots\}$.
- $W = \{\langle n, m \rangle \in \mathbf{N} \times \mathbf{N} : n > m\}$.
- R_1 is the reflexive and symmetric closure of

$$\{\langle \langle n, n-1 \rangle, \langle n+1, 0 \rangle \rangle : n \in \{1, \ldots\}\}.$$

- R_2 is the reflexive, transitive, and symmetric closure of

$$\{\langle \langle n, i \rangle, \langle n, j \rangle \rangle : i, j < n\}.$$

- $c_1^{S(\langle n,m \rangle)} = m$, $c_2^{S(\langle n,m \rangle)} = n$.
- $c^{S(\langle n,m \rangle)} = m$.

- Let $f = \langle f_1, f_2 \rangle$ be a bijection from W onto $\mathbf{N} \times \mathbf{N}$.

$$c_{11}^{S(w)} = f_1(w),\ c_{21}^{S(w)} = f_2(w)\},\ c_{31}^{S(w)} = f_1(w) + f_2(w),$$

$$c_{12}^{S(w)} = f_1(w),\ c_{22}^{S(w)} = f_2(w),\ c_{32}^{S(w)} = f_1(w) \cdot f_2(w).$$

Then $0 \models \Sigma$. ❏

8 Quantifying in with one variable

We have shown above that two types of rather weak fragments of first order common knowledge logics (e.g., their two variable fragments as well as their monadic fragments) are undecidable and mostly even not recursively enumerable. The only natural decidable fragment considered so far turned out to be the one variable fragment. But this is certainly not expressive enough for applications.

It follows from these results that expressive and useful but still decidable fragments require more subtle definitions than just restrictions on the arity of predicates or the number of variables in formulas. In this section we will consider fragments which allow the application of epistemic operators to formulas with one free variable only.

We define the fragment $\mathcal{CL}$ of $\mathcal{CEL}^=$. Let $\varphi \in \mathcal{CEL}^=$ be a formula containing local predicate symbols and global constants only. Then $\varphi \in \mathcal{CL}$ iff φ contains no subformula of the form $\Box\psi$, $\Box \in \{\mathbf{C}, \mathbf{K}_1, \mathbf{K}_2\}$, such that ψ contains more than one free variable. $\mathcal{CL}$ is undecidable since its first order non-epistemic fragment is undecidable. We are now going to develop a criterion which allows to verify in a straightforward manner whether a given fragment of $\mathcal{CL}$ is decidable.

For a set of formulas Γ we denote by $sub_n(\Gamma)$ the closure under negation of the set of all subformulas of formulas in Γ containing precisely n free variables. Without loss of generality we may identify ψ and $\neg\neg\psi$, for any formula ψ; so $sub_n(\varphi)$ is finite.

Let $\Box \in \{\mathbf{C}, \mathbf{K}_1, \mathbf{K}_2\}$. Reserve for any formula $\psi(x) = \Box\varphi(x)$ with one free variable x a unary predicate $P_\psi(x)$ and reserve for any sentence $\psi = \Box\varphi$ a proposional variable p_ψ. For a formula φ we denote by $\overline{\varphi}$ the formula (without epistemic operators) which results when subformulas of the form $\Box\varphi(x)$ and $\Box\varphi$ which are not within the scope of a modal operator are replaced with $P_{\Box\varphi}(x)$ and $p_{\Box\varphi}$, respectively. Denote by $const\varphi$ the set of constants in a formula φ.

DEFINITION 8.1 (type) Fix a formula φ and let x denote a variable which does not occur in φ. Put

$$sub_x(\varphi) = \{\psi(y/x) : \psi(y) \in sub_1(\varphi)\}.$$

A 1-type t for φ is a subset of $sub_x(\varphi)$ such that

- $\psi_1 \wedge \psi_2 \in t$ iff $\psi_1, \psi_2 \in t$, for every $\psi_1 \wedge \psi_2 \in sub_x(\varphi)$.
- $\neg\psi \in t$ iff $\psi \notin t$, for every $\psi \in sub_x(\varphi)$.

A 0-type t for φ is a subset of $sub_0(\varphi)$ such that

- $\psi_1 \wedge \psi_2 \in t$ iff $\psi_1, \psi_2 \in t$, for every $\psi_1 \wedge \psi_2 \in sub_0(\varphi)$.
- $\neg\psi \in t$ iff $\psi \notin t$, for every $\psi \in sub_0(\varphi)$.

For $c \in const(\varphi)$ we call a 1–type $t_c(x)$ for φ and indexed 1–type.

In what follows we shall often not distinguish between the set t and the conjunction $\bigwedge t$.

DEFINITION 8.2 (world candidate) Let T be a set of 1-types for φ, $T^{const} = \{t_c : c \in const\varphi\} \subseteq T$ a set of indexed 1–types for φ, and let Φ be a 0-type for φ. Then $\mathcal{T} = \langle T, T^{const}, \Phi \rangle$ is called a world candidate for φ.

THEOREM 8.3 *Let $\Sigma \subset \mathcal{CL}$. Suppose there is a procedure capable to decide for every $\varphi \in \Sigma$ and every φ-type $\mathcal{T} = \langle T, T^{const}, \Phi \rangle$, whether the formula*

$$\chi(\varphi, \mathcal{T}) = \bigwedge \overline{\Phi} \wedge \bigwedge \{\exists x \; \overline{t}(x) : t \in T\} \wedge \forall x \; \bigvee \{\overline{t}(x) : t \in T\} \wedge \bigwedge \{\overline{t_c}(c) : c \in const\varphi\}$$

is satisfiable. Then, for any standard first order common knowledge logic L, its fragment $\Sigma \cap L$ is decidable.

Observe that $\chi(\varphi, \mathcal{T})$ is a first order formula without epistemic operators. Thus we have reduced the decision problem for Σ to a decision problem for fragments of first order logics. The proof of this result, however, is rather technical and requires various new techniques from modal logic. We omit it here. It is similar to decidability proof for modal description logics in [16].

The following Corollary follows immediately from the decidability of the monadic fragment and the two variable fragment of first order predicate logic.

COROLLARY 8.4 *Let L be a standard first order common knowledge logic.*

- *Let $\Sigma \subseteq \mathcal{CL}$ be the fragment of $\mathcal{CL}$ containing monadic predicates and constants only. Then $\Sigma \cap L$ is decidable.*
- *Let $\Sigma \subseteq \mathcal{CL}$ be the fragment of $\mathcal{CL}$ containing formulas with two variables only. Then $\Sigma \cap L$ is decidable.*

Rather useful fragments of $\mathcal{CL}$ to which Corollary 8.4 applies are the common knowledge description logics. From the technical viewpoint description logics can be characterized as variable free fragments of first order predicate logics (sometimes extended by means of fixpoint operators). They originate from practical knowledge representation systems which, in turn, can be traced back to semantic networks and frames, see e.g.,[4]. An application domain is represented in terms of concepts (alias unary predicates), roles (alias binary predicates), and object names (alias constants). The expressive power of a description logic depends on the concept and role constructors available in the language; examples are conjunction and negation of concepts, and composition, union, and reflexive transitive closure of roles. Here we are going to consider the basic description logic $\mathcal{ALC}$ only. We refer the reader to [5] for more information about description logics.

DEFINITION 8.5 (language) The language of common knowledge description logic is based on a list of

- concept names $C_0, C_1, \ldots$,
- a list of role names $P_0, P_1, \ldots$,
- a list of object names $a_0, a_1, \ldots$,
- and the modal operators $\mathbf{K}_1, \mathbf{K}_2$, and $\mathbf{C}$.

Starting from these we can form concepts and formulas using the following constructors. Firstly we define concepts: every concept name is a concept and

- if C,D are concepts, then $\top$, $C \wedge D$, $\neg C$ are concepts,
- if C is a concept and R is a role name, then $\exists R.C$ is a concept,
- if C is a concepts, then $\mathbf{K}_1 C$, $\mathbf{K}_2 C$, $\mathbf{C}D$ are concepts.

Atomic formulas are expressions of the form $\top$, $C = D$, $a : C$, aRb, where C and D are concepts, R is a role name and a, b are object names. Every atomic formula is a formula and if φ and ψ are formulas then so are $\varphi \wedge \psi$, $\neg\varphi$, $\mathbf{K}_1\varphi$, $\mathbf{K}_2\varphi$, and $\mathbf{C}\varphi$.

The set of all formulas is denoted by $\mathcal{ALCM}$. The intended meaning of the concepts and formulas will be clear from the following translation T into the language of first order common knowledge logic.

Fix different variables x, y. The translation C^T of a concept C is a formula with not more than one free variable. It is defined inductively as follows:

$$\begin{aligned}
C_i^T &= P_i(x) \\
\top^T &= \exists z(z = z) \\
(\exists R.C)^T &= \exists y(R(x,y) \wedge C^T[y/x]) \\
(C \wedge D)^T &= C^T \wedge D^T \\
(\neg C)^T &= \neg C^T \\
(\mathbf{K}_i C)^T &= \mathbf{K}_i C^T \\
(\mathbf{C}C)^T &= \mathbf{C}C^T
\end{aligned}$$

The translation φ^T of a formula φ is a closed formula. It is defined inductively as follows:

$$\begin{array}{rcl} (C = D)^T & = & \forall x(C^T \Leftrightarrow D^T) \\ (a : C)^T & = & C^T[a/x] \\ (aR_ib)^T & = & R_i(a,b) \\ (\mathbf{K}_i\varphi)^T & = & \mathbf{K}_i\varphi^T \\ (\mathbf{C}\varphi)^T & = & \mathbf{C}\varphi^T \end{array}$$

Using Theorem 8.3 the following result is easily shown.

THEOREM 8.6 *Let L be a standard first order common knowledge logic. Then*

$$\{\varphi^T : \varphi \in \mathcal{ALCM}\} \cap L$$

is decidable.

Various decidability results for epistemic description logics are covered by this Theorem, e.g. the decidability results of [1], [8], [13], and some results of [17].

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