Multi-dimensional description logics

Frank Wolter
Institut für Informatik
Universität Leipzig
Augustus-Platz 10-11
04109 Leipzig, Germany
e-mail: wolter@informatik.uni-leipzig.de

Michael Zakharyaschev
Institute of Applied Mathematics
Russian Academy of Sciences
Miusskaya Square 4
125047 Moscow, Russia
e-mail: mz@spp.keldysh.ru

Abstract

In this paper, we construct a new concept description language intended for representing dynamic and intensional knowledge. The most important feature distinguishing this language from its predecessors in the literature is that it allows applications of modal operators to *all* kinds of syntactic terms: concepts, roles and formulas. Moreover, the language may contain both local (i.e., state-dependent) and global (i.e., state-independent) concepts, roles and objects. All this provides us with the most complete and natural means for reflecting the dynamic and intensional behaviour of application domains. We construct a satisfiability checking (mosaic-type) algorithm for this language (based on $\mathcal{ALC}$) in (i) arbitrary multimodal frames, (ii) frames with universal accessibility relations (for knowledge) and (iii) frames with transitive, symmetrical and euclidean relations (for beliefs). On the other hand, it is shown that the satisfaction problem becomes undecidable if the underlying frames are arbitrary linear orders or the language contains the common knowledge operator for $n \geq 2$ agents.

1 Introduction

Description logics are often characterized as logic-based formalisms intended for representing knowledge about concept hierarchies and supplied with effective reasoning procedures and a Tarski-style declarative semantics. A standard example is the description logic $\mathcal{ALC}$ (see [Schmidt-Schauß and Smolka, 1991]) in the syntax of which the "definition" above can be represented as follows:

$$\begin{gathered}\mathsf{Description\_logic} = \mathsf{Knowledge\_representation\_language} \wedge \\ \mathsf{Logic} \wedge \exists \mathsf{is\_decided\_by}.\mathsf{Algorithm} \wedge \exists \mathsf{has}.\mathsf{Tarski\_semantics}, \\ \mathcal{ALC} : \mathsf{Description\_logic}\end{gathered}$$

(here Description_logic, Knowledge_representation_language, Logic, Algorithm, Tarski_semantics are concept names (unary predicates), is_decided_by and has are role names (binary predicates) and $\mathcal{ALC}$ is an object name (individual constant)).

Created in the 1980s as a direct successor of semantic networks and Minsky frames, description logic has found numerous applications and given rise to a rich family of languages (see e.g. [Brachman and Schmolze, 1985; Donini *et al.*, 1996]). But as the application areas are becoming more and more sophisticated, new, more expressive description logics are being called for. Sometimes it is possible to comply with the application demands by enriching a "standard" description language with new constructs and retaining basically the same semantical paradigm. E.g., De Giacomo and Lenzerini [1996] extend $\mathcal{ALC}$ by providing means to form the union, composition, inversion, transitive reflexive closure of roles and to use the number restrictions for quantification over roles; Baader and Hanschke [1991] add concrete domains to $\mathcal{ALC}$. However, some constructs require more drastic changes in the standard semantics. This happens, for instance, when one has to take into account various dynamic aspects of knowledge representation, say time- or agent-dependence of knowledge like

$$\begin{gathered}\mathsf{Received} = \mathsf{Mail} \wedge \langle \textit{sometime in the past} \rangle \exists \mathsf{is\_in}.\mathsf{Mailbox}, \\ \textit{John} : \exists [\textit{always}] \mathsf{loves}.\mathsf{Woman}, \\ [\textit{John believes}] \langle \textit{eventually} \rangle (\exists \mathsf{loves}.\top = \top),\end{gathered}$$

i.e., a received mail has been put into the mailbox some time ago, John will always love the same woman, and John believes that sometime in the future everybody will love somebody.

Several approaches to the design of "dynamic" description logics were developed in the 1990s (see e.g. [Schmiedel, 1990; Schild, 1993; Baader and Ohlbach, 1995; Baader and Laux, 1995; Donini *et al.*, 1992; Wolter and Zakharyaschev, 1998a; 1998b; 1998c]), and all of them share one important feature: their models become multi-dimensional in the sense that besides the usual "object dimension" they may contain a time axis, possible worlds or states for beliefs or actions, etc.

Perhaps the most general multi-dimensional perspective was proposed by Baader and Ohlbach [1993; 1995]. Roughly, each dimension (object, time, belief, etc.) is represented by a set D_i (of objects, moments of time, possible worlds, etc.), concepts are interpreted as subsets of the Cartesian product $\prod_{i=1}^{n} D_i$ and roles of dimension i as binary relations between n-tuples that may differ only in the ith coordinate. And one can quantify over roles not only concepts, but also roles themselves and concept equations. However, the constructed language turned out to be too expressive. At least no sound and complete reasoning procedure for it has appeared (Baader and Ohlbach provide only a sound satisfiability algorithm for a restricted fragment of their language). Moreover, under the natural assumption that some dimensions may be "independent" the language becomes undecidable.[1]

Trying to simplify this semantics, Baader and Laux [1995] noticed that different dimensions may have a different status. For instance, time should probably be the same for all objects inhabiting the object dimension of our knowledge base. This observation led to a somewhat more transparent semantics: models now consist of worlds (or states) which represent—in terms of some standard description logic—the "current state of affairs"; these worlds may change with time passing by or under certain actions, or they may have a number of alternative worlds reflecting the beliefs of agents, and the connection between concepts and roles from different worlds is described by means of the corresponding temporal, dynamic, epistemic, or some other "modal" operators.

There are several "degrees of freedom" within this semantical paradigm.

1. The worlds in models may have arbitrary, or expanding (with respect to the accessibility relation between worlds), or constant domains. Of course, the choice depends on the application we deal with. However, from the technical point of view the most important is the *constant domain assumption*: as was shown in [Wolter and Zakharyaschev, 1998b], if the satisfaction problem is decidable in models with constant domains then it is decidable in models with varying or expanding domains as well. This is the reason why in this paper we adopt the constant domain assumption.

2. The concept, role and object names of the underlying description language may be *local* or *global*. Global names have the same extensions in all worlds, while local ones may have different extensions. (For example, an agent A may regard the role loves to be local, while the role believes to be global.) In principle, we may need both kinds of names. However technically, local object names present no difficulty as compared with global ones, and global concepts are expressible via local concepts and the modal operators.

3. As we saw above, in general one may need modal operators applicable to concepts, roles and formulas.

And finally, depending on the application domain we may choose between various kinds of modal operators (e.g. temporal, epistemic, action, etc.), the corresponding accessibility relations (say, linear for time, universal for knowledge, arbitrary for actions), and between the underlying pure description logics.

The main objective of this paper is to analyze a number of basic multi-dimensional modal description logics based on $\mathcal{ALC}$ and having the most expressive combination of the listed parameters. In particular, we show that the satisfaction problem (and so many other reasoning problems as well) for the logics with modal operators applicable to arbitrary concepts, (local and global) roles and formulas is *decidable* in the class of all (multi-modal) frames, in the class of universal frames (corresponding to the modality "agent A knows") and in the class of transitive, symmetrical and euclidean frames (corresponding to the modality "agent A believes").

Multi-dimensional modal description logics of such a great expressive power have never been considered in the literature. Languages with modal operators applicable only to axioms were studied by Finger and Gabbay [1992] and Laux [1994]; Schild [1993] allows applications of temporal operators only to concepts. Baader and Laux [1995] prove the decidability of the satisfaction problem for $\mathcal{ALC}$ extended with modal operators applicable to concepts and axioms, but only in the class of arbitrary frames and under the expanding domain assumption. Wolter and Zakharyaschev [1998a; 1998b; 1998c] have obtained a series of decidability results for the most important epistemic, temporal and dynamic description logics (based on the description logic $\mathcal{CIQ}$ of [De Giacomo and Lenzerini, 1996]) under the constant domain assumption and with modal operators applicable to both concept and formulas.

However, the computational behaviour of the modalized roles (i.e., binary predicates) has remained unclear. It should be emphasized that this problem is not of only technical interest. Modalized roles are really required for expressing the dynamic features of roles while passing from one state to another (which is usually much more difficult than to reflect the dynamic behaviour of concepts). For instance, to describe the class of people always voting for the same party we can use the axiom

$$\mathsf{Faithful\_voter} = \mathsf{Voter} \wedge \exists [always] \mathsf{votes.Party}.$$

(By swapping $\exists$ and $[always]$ we get the class of people always voting for *some* party.)

The price we have to pay for this extra expressive power is that only a limited number of logics in this language enjoy decidability. We show, for instance, that the satisfaction problem in linear frames or in universal frames with the common knowledge operator for $n \geq 2$ agents is undecidable (but it becomes decidable if the language contains neither global nor modalized roles).

To simplify presentation, we will be considering first description logics with only one modal operator and then generalize the obtained results to systems of multimodal description logic.

2 The language and its models

The primitive symbols of the *modal description language* $\mathcal{ALC}_{\mathcal{M}}$ we deal with in this paper are: *concept names*

[1] Franz Baader has kindly informed us that the language is undecidable without this assumption as well.

$C_0, C_1, \dots,$ *role names* $R_0, R_1, \dots,$ and *object names* $a_0, a_1, \dots$. Starting from these we construct compound *concepts* and *roles* in the following way. Let R be a role, C, D concepts, and let $\Box$ and $\Diamond$ be the (dual) modal "necessity" and "possibility" operators, respectively. Then $\Diamond R$, $\Box R$ are roles, and $\top$, $C \land D$, $\neg C$, $\Diamond C$, $\exists R.C$ are concepts. *Atomic formulas* are expressions of the form $\top$, $C = D$, aRb, $a : C$, where a, b are object names. If φ and ψ are *formulas* then so are $\Diamond\varphi$, $\neg\varphi$, and $\varphi \land \psi$.

The intended semantics of $\mathcal{ALC}_{\mathcal{M}}$ is a natural combination of the standard Tarski-type semantics for the description part of $\mathcal{ALC}_{\mathcal{M}}$ and the Kripke-type (possible world) semantics for the modal part.

Definition 1 (model). An $\mathcal{ALC}_{\mathcal{M}}$-*model* based on a frame $\mathfrak{G} = \langle W, \lhd \rangle$[2] is a pair $\mathfrak{M} = \langle \mathfrak{G}, I \rangle$ in which I is a function associating with each $w \in W$ an $\mathcal{ALC}$-model

$$I(w) = \left\langle \Delta, R_0^{I,w}, \dots, C_0^{I,w}, \dots, a_0^{I,w}, \dots \right\rangle,$$

where Δ is a non-empty set, the *domain* of $\mathfrak{M}$, $R_i^{I,w}$ are binary relations on Δ, $C_i^{I,w} \subseteq \Delta$, and $a_i^{I,w}$ are objects in Δ such that $a_i^{I,u} = a_i^{I,v}$, for any $u, v \in W$. The *values* $C^{I,w}$, $R^{I,w}$ of a concept C and a role R in a world $w \in W$, and the *truth-relation* $(\mathfrak{M}, w) \models \varphi$ (or simply $w \models \varphi$) are defined inductively as follows:

1. $x(\Diamond R)^{I,w} y$ iff $\exists v \rhd w \; xR^{I,v}y$;
2. $x(\Box R)^{I,w} y$ iff $\forall v \rhd w \; xR^{I,v}y$;
3. $(C \land D)^{I,w} = C^{I,w} \cap D^{I,w}$;
4. $(\neg C)^{I,w} = \Delta - C^{I,w}$;
5. $x \in (\Diamond C)^{I,w}$ iff $\exists v \rhd w \; x \in C^{I,v}$;
6. $x \in (\exists R.C)^{I,w}$ iff $\exists y \in C^{I,w} \; xR^{I,w}y$;
7. $w \models C = D$ iff $C^{I,w} = D^{I,w}$;
8. $w \models a : C$ iff $a^{I,w} \in C^{I,w}$;
9. $w \models aRb$ iff $a^{I,w} R^{I,w} b^{I,w}$;
10. $w \models \Diamond\varphi$ iff $\exists v \rhd w \; v \models \varphi$;
11. $w \models \varphi \land \psi$ iff $w \models \varphi$ and $w \models \psi$;
12. $w \models \neg\varphi$ iff $w \not\models \varphi$.

A formula φ is *satisfiable* if there is a model $\mathfrak{M}$ and a world w in $\mathfrak{M}$ such that $w \models \varphi$.

Since many reasoning tasks are reducible to the satisfaction problem for formulas (see e.g. [Donini *et al.*, 1996] and [Wolter and Zakharyaschev, 1998b]), in this paper we focus attention only on the latter. Our first aim is to show that the satisfaction problem is decidable in the class of all $\mathcal{ALC}_{\mathcal{M}}$-models.

By the *modal depth* $md(\varphi)$ of a formula φ we mean the length of the longest chain of nested modal operators in φ (including those in the concepts and roles occurring in φ). It is well known from modal logic (see e.g. [Chagrov and Zakharyaschev, 1997]) that every satisfiable purely modal formula φ can be satisfied in a finite intransitive tree of depth $\le md(\varphi)$. We remind the reader that a frame $\mathfrak{G} = \langle W, \lhd \rangle$ is called a *tree* if (i) $\mathfrak{G}$ is *rooted*, i.e., there is $w_0 \in W$ (a *root* of $\mathfrak{G}$) such that $w_0 \lhd^* w$ for every $w \in W$, where $\lhd^*$ is the transitive and reflexive closure of $\lhd$, and (ii) for every $w \in W$, the set $\{v \in W : v \lhd^* w\}$ is finite and linearly ordered by $\lhd^*$. The *depth* of a tree is the length of its longest branch. And by the *co-depth* of w we mean the number of worlds in the chain $\{v \in W : v \lhd^* w\}$. A tree $\mathfrak{G}$ is *intransitive* if every world v in $\mathfrak{G}$, save its root, has precisely one predecessor, i.e., $|\{u \in W : u \lhd v\}| = 1$, and the root w_0 is *irreflexive*, i.e., $\neg w_0 \lhd w_0$. Using the standard technique of modal logic one can prove the following lemma.

Lemma 2. *Every satisfiable formula is satisfied in a model based on an intransitive tree of depth* $\le md(\varphi)$ *(but possibly with infinitely many branches).*

3 Quasimodels

Fix an $\mathcal{ALC}_{\mathcal{M}}$-formula φ. Let $ob\varphi$ be the set of all object names in φ, and by $con\varphi$, $rol\varphi$ and $sub\varphi$ we denote the sets of all concepts, roles, and subformulas occurring in φ, respectively.

In general, $\mathcal{ALC}_{\mathcal{M}}$-models are rather complex structures with rich interactions between worlds, concepts and roles. That is why standard methods of establishing decidability (say, filtration) do not go through for them. Our idea is to factorize the models modulo φ in such a way that the resulting structures—we will call them *quasimodels*—can be constructed from a finite number of relatively small finite pieces called *blocks*.

Definition 3 (types). A *concept type* for φ is a subset t of $con\varphi$ such that

- $C \land D \in t$ iff $C, D \in t$, for every $C \land D \in con\varphi$;
- $\neg C \in t$ iff $C \notin t$, for every $\neg C \in con\varphi$.

A *named concept type* is the pair $t_a = \langle t, a \rangle$, where t is a concept type and $a \in ob\varphi$. A *formula type* and a *named formula type* for φ are defined analogously as saturated subsets of $sub\varphi$. Finally, by a *type* for φ we mean the pair $\tau = (t, \Xi)$, where t is a concept type and Ξ a formula type for φ; $\tau_a = (t_a, \Xi_a)$ is a *named type* for φ.

To simplify notation we write $C \in \tau$ and $\psi \in \tau$ whenever $\tau = (t, \Xi)$, $C \in t$ and $\psi \in \Xi$. Two types $\tau_1 = (t_1, \Xi_1)$ and $\tau_2 = (t_2, \Xi_2)$ are *formula-equivalent* if $\Xi_1 = \Xi_2$.

Let $\mathfrak{M} = \langle \mathfrak{G}, I \rangle$ be a model over Δ and $\mathfrak{G} = \langle W, \lhd \rangle$ an intransitive tree of depth $\le md(\varphi)$. Without loss of generality we may assume also that $ob\varphi \subseteq \Delta$ (and $a_i^{I,w} = a_i$). For every pair $x \in \Delta$, $w \in W$, let $\tau(x, w) = (t(x, w), \Xi(w))$, where

$$t(x, w) = \{C \in con\varphi : x \in C^{I,w}\},$$
$$\Xi(w) = \{\psi \in sub\varphi : w \models \psi\}.$$

Clearly, $\tau(x, w)$ is a type for φ. The set of (labelled) types $\tau(x, w)$, $w \in W$, with the relation $<_x$ defined by $\tau(x, u) <_x \tau(x, v)$ iff $u \lhd v$ is a tree isomorphic to $\mathfrak{G}$. But modulo φ only a finite part of this tree is enough to represent all the essential information it contains.

[2] W is a non-empty set of *worlds* and $\lhd$ a binary *accessibility* relation on W.

Definition 4 (type tree). By a *type tree* for φ we mean a structure of the form $\mathfrak{T} = \langle T, < \rangle$, where T is a finite set of labelled types for φ (so that one type may have many occurrences in T) and $<$ an intransitive tree order on T such that

(a) for all $\tau \in T$ and $\Diamond C \in con\varphi$, we have $\Diamond C \in \tau$ iff $\exists \tau' > \tau \; C \in \tau'$;

(b) for all $\tau \in T$ and $\Diamond \psi \in sub\varphi$, we have $\Diamond \psi \in \tau$ iff $\exists \tau' > \tau \; \psi \in \tau'$;

(c) $\mathfrak{T}$ is of depth $\leq md(\varphi)$;

(d) if $\tau < \tau'$, $\tau < \tau''$ and $\tau' \neq \tau''$ then the subtrees of $\mathfrak{T}$ generated by τ' and τ'' are not isomorphic.

It should be clear that there exist at most $N_d(\varphi)$ pairwise non-isomorphic type trees of depth d, where

$$N_1(\varphi) = 2^{|con\varphi|} \cdot 2^{|sub\varphi|},$$

$$N_{n+1}(\varphi) = 2^{|con\varphi|} \cdot 2^{|sub\varphi|} \cdot 2^{N_n(\varphi)}.$$

So the number of types in each type tree for φ does not exceed $\sharp(\varphi) = (N_{md(\varphi)}(\varphi))^{md(\varphi)}$.

Definition 5 (type forest). A *type forest of depth d* over Δ is a set $\mathfrak{F} = \{\mathfrak{T}_x : x \in \Delta\}$, where all $\mathfrak{T}_x = \langle T_x, <_x \rangle$ are type trees for φ of the same depth d and $\mathfrak{T}_a$, for every $a \in \Delta \cap ob\varphi$, consists of only named types of the form τ_a.

To represent worlds in models with their inner complex structure we require the following definition.

Definition 6 (run). A *run of co-depth d* through a type forest $\mathfrak{F}$ over Δ is a pair of the form

$$r = \langle \Delta_r, \{R_r : R \in rol\varphi\} \rangle$$

in which Δ_r contains precisely one type $r(x) \in T_x$ of co-depth d for every $x \in \Delta$ (so that $\Delta_r = \{r(x) : x \in \Delta\}$) and $R_r \subseteq \Delta_r \times \Delta_r$ such that:

(e) all types in Δ_r are formula-equivalent to each other;

(f) $\exists R.C \in r(x)$ iff $\exists y \in \Delta \; (r(x) R_r r(y) \; \& \; C \in r(y))$, for every $\exists R.C \in con\varphi$;

(g) $C = D \in r(x)$ iff $\forall y \in \Delta \; (C \in r(y) \Leftrightarrow D \in r(y))$, for every $C = D \in sub\varphi$;

(h) $a : C \in r(x)$ iff $C \in r(a)$, for every $a : C \in sub\varphi$ provided that $a \in \Delta$;

(i) $aRb \in r(x)$ iff $r(a) R_r r(b)$, for every $aRb \in sub\varphi$ provided that $a, b \in \Delta$.

If only the ($\Leftarrow$)-part of (f) holds, we call r a *weak run* of co-depth d. And if a weak run r satisfies (f) for some particular x in Δ, then r is called a *weak x-saturated run* of co-depth d. Instead of $\psi \in r(x)$ we will write $r \models \psi$.

Models as a whole are represented in the form of quasimodels.

Definition 7 (quasimodel). A triple $\mathfrak{m} = \langle \mathfrak{F}, \mathfrak{R}, \lhd \rangle$ is called a *quasimodel* for φ if $\mathfrak{F}$ is a type forest of depth $m \leq md(\varphi)$ for φ over some $\Delta \supseteq ob\varphi$, $\mathfrak{R}$ a set of runs through $\mathfrak{F}$ and $\lhd$ is an intransitive tree order on $\mathfrak{R}$ such that the following conditions hold:

(j) for every $d \leq m$, the set $\mathfrak{R}^d$ of runs of co-depth d in $\mathfrak{R}$ is non-empty;

(k) for any $r, r' \in \mathfrak{R}$, if $r \lhd r'$ then $r(x) <_x r'(x)$ for all $x \in \Delta$;

(l) for all $r \in \mathfrak{R}^d$, $x \in \Delta$, and $\tau \in T_x$, if $r(x) <_x \tau$ then there is $r' \in \mathfrak{R}^{d+1}$ such that $r'(x) = \tau$ and $r \lhd r'$;

(m) for all $x, y \in \Delta$, $d \leq m$, $\Diamond R \in rol\varphi$, and $r \in \mathfrak{R}^d$, we have $r(x)(\Diamond R)_r r(y)$ iff $\exists r' \rhd r \; r'(x) R_{r'} r'(y)$;

(n) for all $x, y \in \Delta$, $d \leq m$, $\Box R \in rol\varphi$, and $r \in \mathfrak{R}^d$, we have $r(x)(\Box R)_r r(y)$ iff $\forall r' \rhd r \; r'(x) R_{r'} r'(y)$.

We say $\mathfrak{m}$ *satisfies* φ if $r \models \varphi$ for some $r \in \mathfrak{R}$.

To reconstruct the model $\mathfrak{M} = \langle \mathfrak{G}, I \rangle$ factorized in a quasimodel $\mathfrak{m} = \langle \mathfrak{F}, \mathfrak{R}, \lhd \rangle$ over domain Δ, one can take $\mathfrak{G} = \langle W, \lhd \rangle$, $W = \bigcup \mathfrak{R}$, and put $x R_i^{I,r} y$ iff $r(x)(R_i)_r r(y)$, $x \in C_i^{I,r}$ iff $C_i \in r(x)$, $a_i^{I,r} = a_i$,

$$I(r) = \langle \Delta, R_0^{I,r}, \ldots, C_0^{I,r}, \ldots, a_0^{I,r}, \ldots \rangle,$$

Thus we obtain:

Theorem 8. *A formula φ is satisfiable iff φ is satisfied in some quasimodel for φ.*

Let $\mathfrak{m} = \langle \mathfrak{F}, \mathfrak{R}, \lhd \rangle$ be a quasimodel over Δ, $x \in \Delta$, $R \in rol\varphi$ and let $R = \mathsf{M} R_i$ for some (possibly empty) string M of $\Diamond$ and $\Box$, R_i a role name. Consider $\mathfrak{T}_x = \langle T_x, <_x \rangle$ as a usual Kripke frame. If $(\mathfrak{T}_x, r(x)) \models \mathsf{M}\bot$, $r \in \mathfrak{R}$, then we say that R is *r-universal*. This name is explained by the fact that if R is r-universal then we have $R_r = \Delta_r \times \Delta_r$, which can be easily established by induction on the length of M. Using the standard unravelling technique of modal logic (see e.g. [Chagrov and Zakharyaschev, 1997]) one can prove

Lemma 9. *Every satisfiable φ is satisfied in a quasimodel $\langle \mathfrak{F}, \mathfrak{R}, \lhd \rangle$ for φ in which the set of pairs $\langle x, y \rangle$ such that $x, y \notin ob\varphi$ and $r(x) R_r r(y)$ for some R that is not r-universal, $r \in \mathfrak{R}$, is an intransitive forest order on the set of objects involved in this relation.*

4 Satisfiability checking

We are in a position now to show that a formula φ is satisfiable iff one can construct a (possibly infinite) quasimodel satisfying φ out of a finite set of finite pattern blocks.

Definition 10 (block). Let $\mathfrak{F}$ be a type forest for φ of depth $m \leq md(\varphi)$ over a finite Δ which is disjoint from $ob\varphi$, x an object in Δ, $\mathfrak{R}$ a set of weak x-saturated runs through $\mathfrak{F}$ such that the set of pairs $\langle x, y \rangle$ with $r(x) R_r r(y)$ for some R that is not r-universal, $r \in \mathfrak{R}$, is an intransitive tree order on Δ with root x, and let $\lhd$ be an intransitive tree order on $\mathfrak{R}$. We say $\langle \mathfrak{F}, \mathfrak{R}, \lhd \rangle$ is a *$\mathfrak{T}_x$-block* for φ if it satisfies conditions (j)–(n).

Definition 11 (kernel block). A *kernel block* over $ob\varphi \neq \emptyset$ is a structure of the form $\langle \mathfrak{F}, \mathfrak{R}, \lhd \rangle$ in which $\mathfrak{F}$ is a type forest over $ob\varphi$ of depth $m \leq md(\varphi)$ (it contains only type trees named by the elements of $ob\varphi$), $\mathfrak{R}$ a set of weak runs through $\mathfrak{F}$ and $\lhd$ an intransitive tree order on $\mathfrak{R}$ satisfying (j)–(n).

Definition 12 (satisfying set). A set of blocks $\mathcal{S}$ for φ is called a *satisfying set* for φ if

(o) $\mathcal{S}$ contains one kernel block for φ whenever $ob\varphi \neq \emptyset$;

(p) in every block $\langle \mathfrak{F}, \mathfrak{R}, \lhd \rangle \in \mathcal{S}$ there is $r \in \mathfrak{R}$ such that $r \models \varphi$;

(q) for every $\langle \mathfrak{F}, \mathfrak{R}, \lhd \rangle$ in $\mathcal{S}$ and every $\mathfrak{T}_x \in \mathfrak{F}$, there is precisely one $\mathfrak{T}_x$-block in $\mathcal{S}$.

Theorem 13. *A formula φ is satisfiable iff there is a satisfying set for φ the domain of each (non-kernel) block in which contains at most $\sharp(\varphi) \cdot |con\varphi| \cdot (md(\varphi) + 1) + 1$ objects.*

As an immediate consequence we obtain:

Theorem 14. *The satisfaction problem for $\mathcal{ALC}_{\mathcal{M}}$-formulas is decidable.*

So far we were considering satisfiability in *arbitrary* $\mathcal{ALC}_{\mathcal{M}}$-models. However, various specializations of the modal operators may impose different restrictions on the structure of underlying frames in our models. For instance, if we understand $\Box$ as "it is known", we may need frames that are transitive, reflexive and symmetrical, i.e., **S5**-frames in the modal logic terminology, and if $\Box$ is intended to stand for "it is believed", then we may need **KD45**-frames which have the form of an **S5**-frame possibly with one irreflexive predecessor.

It is not hard to adopt the developed technique to prove the following:

Theorem 15. *There is an algorithm which is capable of deciding, given an arbitrary $\mathcal{ALC}_{\mathcal{M}}$-formula φ, whether φ is satisfiable in an $\mathcal{ALC}_{\mathcal{M}}$-model based upon (i) an* **S5***-frame or (ii) a* **KD45***-frame.*

In some applications we may need $\mathcal{ALC}_{\mathcal{M}}$-models with *global roles*, i.e., roles R which are interpreted by the same binary relation in every world of a model. In quasi-models, we can reflect this by requiring that $r(x)R_r r(y)$ implies $r'(x)R_{r'}r'(y)$ for all $x, y \in \Delta$, $r, r' \in \mathfrak{R}$; in other words, global roles correspond to binary relations between type trees. By a straightforward modification of the proof of Theorem 14 one can show the following:

Theorem 16. *There is an algorithm which is capable of deciding, given an arbitrary $\mathcal{ALC}_{\mathcal{M}}$-formula φ with global roles, whether φ is satisfiable in an $\mathcal{ALC}_{\mathcal{M}}$-model based upon (i) an arbitrary frame, (ii) an* **S5***-frame or (iii) a* **KD45***-frame.*

When dealing with intensional knowledge, one usually needs one modal operator $\Box_i$ for each agent i (meaning that "agent i knows" or "agent i believes"); see e.g. [Fagin *et al.*, 1995] for a discussion of propositional multimodal epistemic logics. Let $\mathcal{ALC}_{\mathcal{M}_n}$ denote the modal description language with n modal operators, so that $\mathcal{ALC}_{\mathcal{M}_n}$-models are based on Kripke frames with n accessibility relations $\lhd_1, \ldots, \lhd_n$. These frames are called n-frames. $\mathbf{S5}_n$-frames and $\mathbf{KD45}_n$-frames are those n-frames all monomodal fragments of which are **S5**-frames and **KD45**-frames, respectively. The developed technique provides a satisfiability checking algorithm for this multimodal case as well.

Theorem 17. *For every $n \geq 1$, there is an algorithm which is capable of deciding, for an arbitrary $\mathcal{ALC}_{\mathcal{M}_n}$-formula φ with global roles, whether φ is satisfiable in an $\mathcal{ALC}_{\mathcal{M}_n}$-model based upon (i) an arbitrary n-frame, (ii) an $\mathbf{S5}_n$-frame or (iii) a $\mathbf{KD45}_n$-frame.*

It would also be of interest to extend the constructed epistemic description language $\mathcal{ALC}_{\mathcal{M}_n}$ with the *common knowledge operator* **C** which is interpreted by the transitive and reflexive closure of the union $\lhd_1 \cup \ldots \cup \lhd_n$. (For various applications of **C** in the analysis of multiagent systems see [Fagin *et al.*, 1995].) Another important kind of modality often used in applications is the temporal operator "always in the future" (or the operators "Since" and "Until") interpreted in linearly ordered sets of worlds (see e.g. [Gabbay *et al.*, 1994]). The satisfaction problem for these languages without global and modalized roles is known to be decidable (see [Wolter and Zakharyaschev, 1998a; 1998b; 1998c]). However, this is not the case for the language constructed in this paper:

Theorem 18. (i) *The satisfaction problem for $\mathcal{ALC}_M$-formulas in linearly ordered transitive frames is undecidable; it is undecidable in $\langle \mathbb{N}, < \rangle$ as well.*

(ii) *The satisfaction problem for the epistemic description formulas with $n \geq 2$ agents and the common knowledge operator is undecidable in the class of $\mathbf{S5}_n$-frames.*

5 Discussion and open problems

This paper makes one more step in the study of concept description languages of high expressive power that are located near the boarder between decidable and undecidable. We have designed a "full" multidimensional modal description language which imposes no restrictions whatsoever on the use of modal operators (they can be applied to all types of syntactic terms: concepts, roles and formulas) and contains both local and global object, concept and role names. Using the mosaic technique we have proved that the satisfaction problem for the formulas of this language (and so many other reasoning tasks as well) is decidable in some important classes of models. (Actually, this gives a solution to a problem raised by Baader and Ohlbach [1993].) On the other hand, it was shown that the language becomes undecidable when interpreted on temporal structures or augmented with the common knowledge operator.

The obtained results demonstrate a principle possibility of using this highly expressive language in knowledge representation systems. Further investigations are required to make it really applicable. In particular, it would be of interest to answer the following questions:

(1) Do the logics considered above have the finite model property?

Our conjecture is that they do have this property, and so the finite model reasoning in those logics is effective.

(2) What is the complexity of satisfiability checking in these logics?

We only know that the satisfaction problem in all of them is NEXPTIME-hard.

(3) Is it possible to extend the developed technique to transitive frames?

Our method is heavily based on the fact that models of depth $\leq md(\varphi)$ are always enough to satisfy a given formula φ. This is not the case when the accessibility relations are transitive, i.e., satisfy the natural epistemic axiom $\Box\varphi \rightarrow \Box\Box\varphi$.

To increase the language's capacity of expressing the dynamics of relations between individual objects in application domains it would be desirable also

(4) to extend $\mathcal{ALC}_{\mathcal{M}_n}$ with (some of) the booleans operating on roles,

(5) to extend the underlying description logic with new constructs

and, of course, retain decidability.

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