Universität Leipzig
Fakultät für Mathematik und Informatik
Mathematisches Institut

Periods and Algebraic deRham Cohomology

Diplomarbeit
im Studiengang Diplom-Mathematik

Leipzig, vorgelegt von Benjamin Friedrich,
geboren am 22. Juli 1979

Abstract

It is known that the algebraic deRham cohomology group $\mathrm{H}^i_{\mathrm{dR}}(X_0/\mathbb{Q})$ of a nonsingular variety $X_0/\mathbb{Q}$ has the same rank as the rational singular cohomology group $\mathrm{H}^i_{\mathrm{sing}}(X^{\mathrm{an}};\mathbb{Q})$ of the complex manifold X^{an} associated to the base change $X_0 \times_{\mathbb{Q}} \mathbb{C}$. However, we do not have a natural isomorphism $\mathrm{H}^i_{\mathrm{dR}}(X_0/\mathbb{Q}) \cong \mathrm{H}^i_{\mathrm{sing}}(X^{\mathrm{an}};\mathbb{Q})$. Any choice of such an isomorphism produces certain integrals, so called periods, which reveal valuable information about X_0. The aim of this thesis is to explain these classical facts in detail. Based on an approach of Kontsevich [K, pp. 62–64], different definitions of a period are compared and their properties discussed. Finally, the theory is applied to some examples. These examples include a representation of $\zeta(2)$ as a period and a variation of mixed Hodge structures used by Goncharov [G1].

Contents

1 Introduction

The prehistory of Algebraic Topology dates back to Euler, Riemann and Betti, who started the idea of attaching various invariants to a topological space. With his simplicial (co)homology theory, Poincaré was the first to give an instance of what in modern terms we would call a contravariant functor $\mathrm{H}^\bullet$ from the category of (sufficiently nice) topological spaces to the category of cyclic complexes of abelian groups.
Many of such functors have been found so far; the most common examples are the standard cohomology theories (i.e. those satisfying the Eilenberg-Steenrod axioms), which measure quite different phenomena relating to diverse branches of mathematics. It is a beautiful basic fact that all these standard cohomology theories agree (when restricted to an appropriate subcategory).
This does not imply that we cannot hope for more. If the topological space in question enjoys additional structure, one defines more elaborate invariants which take values in an abelian category of higher complexity. For example, Hodge theory gives us a functor from the category of compact Kähler manifolds to the category of cyclic complexes of pure Hodge structures.
In this thesis, we will concentrate on spaces originating from Algebraic Geometry; these may be regarded as spaces carrying an algebraic structure.
Generalizing the concept of deRham theory to "nice" schemes over $\mathbb{Q}$ gives us algebraic deRham cohomology groups where each is nothing but a full $\mathbb{Q}$-lattice inside the $\mathbb{C}$-vector space of the corresponding classical deRham cohomology group. So, after tensoring with $\mathbb{C}$, algebraic deRham cohomology agrees with all the standard cohomology theories with complex coefficients. However, a natural isomorphism between the original $\mathbb{Q}$-vector space and a standard cohomology group with rational coefficients cannot exist.
We will illustrate this phenomenon in the following example (see Example 4.12.1 for details). Let $X^{\mathrm{an}} := \mathbb{C}^\times$ be the complex plane with the point 0 deleted and let t be the standard coordinate on X^{an}. Then the first singular cohomology group of X^{an} is generated by the dual σ^* of the unit circle $\sigma := S^1$

$$\mathrm{H}^1_{\mathrm{sing}}(X^{\mathrm{an}};\mathbb{Q}) = \mathbb{Q}\sigma^* \quad \text{and} \quad \mathrm{H}^1_{\mathrm{sing}}(X^{\mathrm{an}};\mathbb{C}) = \mathbb{C}\sigma^*;$$

while for the first classical deRham cohomology group, we have

$$\mathrm{H}^1_{\mathrm{dR}}(X^{\mathrm{an}};\mathbb{C}) = \mathbb{C}\,\frac{dt}{t}.$$

Under the comparison isomorphism

$$\mathrm{H}^1_{\mathrm{sing}}(X^{\mathrm{an}};\mathbb{C}) \cong \mathrm{H}^1_{\mathrm{dR}}(X^{\mathrm{an}};\mathbb{C})$$

the generator σ^* of $\mathrm{H}^1_{\mathrm{sing}}(X^{\mathrm{an}};\mathbb{C})$ is mapped to

$$\left(\int_{S^1}\frac{dt}{t}\right)^{-1}\frac{dt}{t} = \frac{1}{2\pi i}\,\frac{dt}{t}.$$

If we view X^{an} as the complex manifold associated to the base change to $\mathbb{C}$ of the algebraic variety $X_0 := \operatorname{Spec}\mathbb{Q}[t,t^{-1}]$ over $\mathbb{Q}$, we can also compute the algebraic de Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(X_0/\mathbb{Q})$ of X_0 and embed it into $\mathrm{H}^1_{\mathrm{dR}}(X^{\mathrm{an}};\mathbb{C})$

$$\mathrm{H}^1_{\mathrm{dR}}(X_0/\mathbb{Q}) = \mathbb{Q}\frac{dt}{t} \subset \mathrm{H}^1_{\mathrm{dR}}(X^{\mathrm{an}};\mathbb{C}) = \mathbb{C}\frac{dt}{t}.$$

Thus we get two $\mathbb{Q}$-lattices inside $\mathrm{H}^1_{\mathrm{dR}}(X^{\mathrm{an}};\mathbb{C})$, $\mathrm{H}^1_{\mathrm{sing}}(X^{\mathrm{an}};\mathbb{Q})$ and $\mathrm{H}^1_{\mathrm{dR}}(X_0/\mathbb{Q})$, which do not coincide. In fact, they differ by the factor $2\pi i$ — our first example of what we will call a period. Other examples will produce period numbers like π, $\ln 2$, elliptic integrals, or $\zeta(2)$, which are interesting also from a number theoretical point of view (cf. page 47).
There is some ambiguity about the precise definition of a period; actually we will give four definitions in total:

(i) pairing periods (cf. Definition 5.1.1 on page 43)

(ii) abstract periods (cf. Definition 5.2.1 on page 45)

(iii) naïve periods (cf. Definition 5.3.2 on page 46)

(iv) effective periods (cf. Definition 7.3.1 on page 63)

For X_0 a nonsingular variety over $\mathbb{Q}$, we have a natural pairing between the $i^{\,\mathrm{th}}$ algebraic deRham cohomology of X_0 and the $i^{\,\mathrm{th}}$ singular homology group of the complex manifold X^{an} associated to the base change $X_0 \times_{\mathbb{Q}} \mathbb{C}$

$$\mathrm{H}_i^{\mathrm{sing}}(X^{\mathrm{an}};\mathbb{Q}) \times \mathrm{H}^i_{\mathrm{dR}}(X_0/\mathbb{Q}) \longrightarrow \mathbb{C}.$$

The numbers which can appear in the image of this pairing (or its version for relative cohomology) are called *pairing periods*; this is the most traditional way to define a period.
In [K, p. 62], Kontsevich gives the alternative definition of *effective periods* which does not need algebraic deRham cohomology and, at least conjecturally, gives the set of all periods some extra algebraic structure. We present his ideas in Subsection 7.3.
Abstract periods describe just a variant of Kontsevich's definition. In fact, we have a surjection from the set of effective periods to the set of abstract ones (cf. page 65), which is conjectured to be an isomorphism.
Naïve periods are defined in an elementary way and are used to provide a connection between pairing periods and abstract periods.
In Kontsevich's paper [K, p. 63], it is used that the notion of pairing and abstract period coincide. The aim of this thesis is to show that the following implications hold true (cf. Theorem 7.1.1)

$$\text{abstract period} \Leftrightarrow \text{naïve period} \Rightarrow \text{pairing period}.$$

The thesis is organized as follows. The discussion of the various definitions of a period makes up the principal part of the work filling sections five to seven.
Section two gives an introduction to complex analytic spaces. Additionally, we provide the connection to Algebraic Geometry by defining the associated complex analytic space of a variety.
In **Section three**, we define algebraic deRham cohomology for pairs consisting of a variety and a divisor on it. We also give some working tools for this cohomology.
The aim of **Section four** is to give a comparison theorem (Theorem 4.10.1) which states that algebraic and singular cohomology agree.
In **Section five**, we present the definition of pairing, abstract, and naïve periods and prove some of their properties.
Section six provides some facts about the triangulation of algebraic varieties.

Section seven contains the main result (Theorem 7.1.1) about the implications between the various definitions of a period mentioned above. Furthermore, we give the definition of effective periods which motivated the definition of abstract periods. In the last section, **Section eight**, we consider five examples to give an application of the general theory. Among them is a representation of $\zeta(2)$ as an abstract period and the famous double logarithm variation of mixed Hodge structures used by Goncharov [G1] whose geometric origin is emphasized.

Conventions. By a *variety*, we will always mean a reduced, quasi-projective scheme. We will often deal with a variety X_0 defined over some algebraic extension of $\mathbb{Q}$. As a rule, skipping the subscript 0 will always mean base change to $\mathbb{C}$

$$X := \text{“}X_0 \times_{\mathbb{Q}} \mathbb{C}\text{”} = X_0 \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\mathbb{C}.$$

(An exception is section three, where arbitrary base fields are used.) The complex analytic space associated to X will be denoted by X^{an} (cf. Subsection 2.1).
The sign conventions used throughout this thesis are listed in the appendix.

Acknowledgments. I am greatly indebted to my supervisor Prof. A. Huber-Klawitter for her guidance and her encouragement. I very much appreciated the informal style of our discussions in which she vividly pointed out to me the central ideas of the mathematics involved.
I would also like to thank my fellow students R. Munck, M. Witte and K. Zehmisch who read the manuscript and gave numerous comments which helped to clarify the exposition.

2 The Associated Complex Analytic Space

Let X be a variety over $\mathbb{C}$. The set $|X|$ of closed points of X inherits the Zariski topology. However, we can also equip this set with the standard topology: For smooth X this gives a complex manifold; in general we get a complex analytic space X^{an}. The main reference for this section is [Ha, B.1].

2.1 The Definition of the Associated Complex Analytic Space

We consider an example before giving the general definition of a complex analytic space.

Example 2.1.1. Let $D^n \subset \mathbb{C}^n$ be the polycylinder

$$D^n := \{\underline{z} \in \mathbb{C}^n \mid |z_i| < 1, i = 1, \ldots, n\}$$

and $\mathcal{O}_{D^n}$ the sheaf of holomorphic functions on D^n. For a set of holomorphic functions $f_1, \ldots, f_m \in \Gamma(D^n, \mathcal{O}_{D^n})$ we define

$$\begin{aligned} \mathcal{X}_{D^n} &:= \{\underline{z} \in D^n \mid f_1(\underline{z}) = \ldots = f_m(\underline{z}) = 0\} \\ \mathcal{O}_{\mathcal{X}_{D^n}} &:= \mathcal{O}_{D^n}/(f_1, \ldots, f_m). \end{aligned} \tag{1}$$

The locally ringed space $(\mathcal{X}_{D^n}, \mathcal{O}_{\mathcal{X}_{D^n}})$ from this example is a complex analytic space. In general, complex analytic spaces are obtained by glueing spaces of the form (1).

Definition 2.1.2 (Complex analytic space, [Ha, B.1, p. 438]). *A locally ringed space $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is called* complex analytic *if it is locally (as a locally ringed space) isomorphic to one of the form (1). A morphism of complex analytic spaces is a morphism of locally ringed spaces.*

For any scheme $(X, \mathcal{O}_X)$ of finite type over $\mathbb{C}$ we have an *associated complex analytic space* $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$.

Definition 2.1.3 (Associated complex analytic space, [Ha, B.1, p. 439]). *Assume first that X is affine. We fix an isomorphism*

$$X \cong \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

and then consider the f_i as holomorphic functions on $\mathbb{C}^n$ in order to set

$$\begin{aligned} X^{\mathrm{an}} &:= \{\underline{z} \in \mathbb{C}^n \mid f_1(\underline{z}) = \ldots = f_m(\underline{z}) = 0\} \\ \mathcal{O}_{X^{\mathrm{an}}} &:= \mathcal{O}_{\mathbb{C}^n}/(f_1, \ldots, f_m), \end{aligned}$$

where $\mathcal{O}_{\mathbb{C}^n}$ denotes the sheaf of holomorphic functions on $\mathbb{C}^n$.
For an arbitrary scheme X of finite type over $\mathbb{C}$, we take a covering of X by open affine subsets U_i. The scheme X is obtained by glueing the open sets U_i, so we can use the same glueing data to glue the complex analytic space $(U_i)^{\mathrm{an}}$ into an analytic space X^{an}. This is the associated complex analytic space of X.

This construction is natural and we obtain a functor **an** from the category of schemes of finite type over $\mathbb{C}$ to the category of complex analytic spaces. Note that its restriction to the subcategory of smooth schemes maps into the category of complex manifolds as a consequence of the inverse function theorem (cf. [Gun, Thm. 6, p. 20]).

Example 2.1.4. The complex analytic space associated to complex projective space is again complex projective space, but considered as a complex manifold. To avoid confusion in the subsequent sections, the notation $\mathbb{C}P^n$ will be reserved for *complex projective space in the category of schemes*, whereas we write $\mathbb{C}P^n_{\text{an}}$ for *complex projective space in the category of complex analytic spaces.*

For any scheme X of finite type over $\mathbb{C}$, we have a natural map of locally ringed spaces

$$\phi : X^{\text{an}} \to X \tag{2}$$

which induces the identity on the set of closed points $|X|$ of X. Note that $\phi^*\mathcal{O}_X = \mathcal{O}_{X^{\text{an}}}$.

2.2 Algebraic and Analytic Coherent Sheaves

Let us consider sheaves of $\mathcal{O}_X$-modules. The equality of functors

$$\Gamma(X, ?) = \Gamma(X^{\text{an}}, \phi^{-1}?)$$

gives an equality of their right derived functors

$$\mathrm{H}^i(X; ?) = \mathrm{R}^i\Gamma(X; ?) = \mathrm{R}^i\Gamma(X^{\text{an}}; \phi^{-1}?).$$

Since ϕ^{-1} is an exact functor, the spectral sequence for the composition of the functors ϕ^{-1} and $\Gamma(X^{\text{an}}; ?)$ degenerates and we obtain

$$\mathrm{H}^i(X^{\text{an}}; \phi^{-1}?) = \mathrm{R}^i\Gamma(X^{\text{an}}; ?) \circ \phi^{-1} = \mathrm{R}^i\Gamma(X^{\text{an}}; \phi^{-1}?).$$

Thus the natural map for $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules

$$\phi^{-1}\mathcal{F} \to \phi^*\mathcal{F}$$

gives a natural map of cohomology groups

$$\mathrm{H}^i(X; \mathcal{F}) = \mathrm{H}^i(X^{\text{an}}; \phi^{-1}\mathcal{F}) \to \mathrm{H}^i(X^{\text{an}}; \phi^*\mathcal{F}). \tag{3}$$

Sheaf cohomology behaves particularly nice for coherent sheaves, this notion being defined as follows.

Definition 2.2.1 (Coherent sheaf). *We define a* coherent sheaf $\mathcal{F}$ *on* X *(resp.* X^{an}*) to be a sheaf of* $\mathcal{O}_X$*-modules (resp.* $\mathcal{O}_{X^{\text{an}}}$*-modules) that is Zariski-locally (resp. locally in the standard topology) isomorphic to the cokernel of a morphism of free* $\mathcal{O}_X$*-modules (resp.* $\mathcal{O}_{X^{\text{an}}}$*-modules) of finite rank*

$$\mathcal{O}_U^r \longrightarrow \mathcal{O}_U^s \longrightarrow \mathcal{F}_{|U} \longrightarrow 0, \quad \textit{for} \quad U \subseteq X \ \textit{(resp. } U \subseteq X^{\text{an}}\textit{)} \quad \textit{open}, \tag{4}$$

where $r, s \in \mathbb{N}$.

For sheaves on X this agrees with the definition given in [Ha, II.5, p. 111]. For this alternate definition, we need some notation: If $U = \operatorname{Spec} A$ is an affine variety and M an A-module, we denote by $\widetilde{M}$ the sheaf on U associated to M (i.e. the sheaf associated to the presheaf $V \mapsto \Gamma(V; \mathcal{O}_V) \otimes_A M$ *for* $V \subseteq U$ open, see [Ha, II.5, p. 110]).

Lemma 2.2.2 (cf. [Ha, II.5 Exercise 5.4, p. 124]). *A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is coherent if and only if X can be covered by open affine subsets $U_i = \operatorname{Spec} A_i$ such that $F_{|U_i} \cong \widetilde{M_i}$ for some finitely generated A_i-modules M_i.*

Proof. "if": The A_i's are Noetherian rings. Therefore any finitely generated A_i-module M_i will be finitely presented

$$A_i^r \longrightarrow A_i^s \longrightarrow M_i \longrightarrow 0.$$

Since localization is an exact functor, we get

$$\mathcal{O}_{U_i}^r \longrightarrow \mathcal{O}_{U_i}^s \longrightarrow \widetilde{M_i} \longrightarrow 0,$$

which proves the "if"-part. ■

"only if": W.l.o.g. we may assume that the open subsets $U \subseteq X$ in (4) are affine $U = \operatorname{Spec} A$. Then $\Gamma(U; \mathcal{O}_U) = A$ and $\mathcal{O}_U = \widetilde{A}$. Now the A-module

$$\begin{aligned} M &:= \operatorname{coker}\bigl(\Gamma(U; \mathcal{O}_U^r) \longrightarrow \Gamma(U; \mathcal{O}_U^s)\bigr) \\ &= \operatorname{coker}(A^r \longrightarrow A^s) \end{aligned}$$

is clearly finitely generated. Since

$$A^r \longrightarrow A^s \longrightarrow M \longrightarrow 0$$

gives

$$\widetilde{A}^r \longrightarrow \widetilde{A}^s \longrightarrow \widetilde{M} \longrightarrow 0,$$

we conclude

$$\mathcal{F}_{|U} = \operatorname{coker}(\mathcal{O}_U^r \longrightarrow \mathcal{O}_U^s) = \operatorname{coker}(\widetilde{A}^r \longrightarrow \widetilde{A}^s) = \widetilde{M}.$$

□

As an immediate consequence of the definition of a coherent sheaf $\mathcal{F}$ on X, we see that the sheaf

$$\mathcal{F}_{\text{an}} := \phi^* \mathcal{F}$$

will be coherent as well: If

$$\mathcal{O}_U^r \longrightarrow \mathcal{O}_U^s \longrightarrow \mathcal{F}_{|U} \longrightarrow 0$$

is exact, so is

$$\mathcal{O}_{U^{\text{an}}}^r \longrightarrow \mathcal{O}_{U^{\text{an}}}^s \longrightarrow \phi^* \mathcal{F}_{|U^{\text{an}}} \longrightarrow 0,$$

since ϕ^{-1} is exact and tensoring is a right exact functor.

There is a famous theorem by Serre usually referred to as GAGA, since it is contained in his paper "*Géométrie algébrique et géométrie analytique*" [Se:gaga].

Theorem 2.2.3 (Serre, [Ha, B.2.1, p. 440]). *Let X be a* **projective** *scheme over $\mathbb{C}$. Then the map*

$$\mathcal{F} \mapsto \mathcal{F}_{\text{an}}$$

induces an equivalence between the category of coherent sheaves on X and the category of coherent sheaves on X^{an}. Furthermore, the natural map (3)

$$\mathrm{H}^i(X; \mathcal{F}) \xrightarrow{\sim} \mathrm{H}^i(X^{\text{an}}; \mathcal{F}_{\text{an}})$$

is an isomorphism for all i.

Let us state a corollary of Theorem 2.2.3, which is not included in [Ha].

Corollary 2.2.4. *In the situation of Theorem 2.2.3, we also have a natural isomorphism for hypercohomology for all i*

$$\mathbb{H}^i(X;\mathcal{F}^\bullet) \cong \mathbb{H}^i(X^{\mathrm{an}};\mathcal{F}^\bullet_{\mathrm{an}}),$$

where $\mathcal{F}^\bullet$ is a bounded complex of coherent sheaves on X.

Here we only require the boundary morphisms of $\mathcal{F}^\bullet$ to be morphisms of sheaves of abelian groups. They do not need to be $\mathcal{O}_X$-linear.
Before we begin proving Corollary 2.2.4, we need some homological algebra.

Lemma 2.2.5. *Let $\mathfrak{A}$ be an abelian category and*

$$\mathrm{F}^\bullet[0] \longrightarrow \mathrm{G}^{\bullet,\bullet}, \tag{5}$$

a morphism of double complexes of $\mathfrak{A}$-objects (cf. the appendix), where

- $\mathrm{F}^\bullet[0]$ *is a double complex concentrated in the zeroth row with* $\mathrm{F}^\bullet$ *being a complex vanishing below degree zero, i.e.* $\mathrm{F}^{\mathrm{n}} = 0$ *for $n < 0$, and*
- $\mathrm{G}^{\bullet,\bullet}$ *is a double complex living only in non-negative degrees.*

If for all $q \in \mathbb{Z}$

$$0 \to \mathrm{F}^{\mathrm{q}} \to \mathrm{G}^{0,\mathrm{q}} \to \mathrm{G}^{1,\mathrm{q}} \to \cdots$$

is a resolution of F^{q}*, then the map of total complexes induced by* (5)

$$\mathrm{F}^\bullet \xrightarrow{\sim} \operatorname{tot} \mathrm{G}^{\bullet,\bullet}$$

is a quasi-isomorphism.

Proof. From (5) we obtain a morphism of spectral sequences

$$\begin{array}{ccc} \mathrm{h}^p_{\mathrm{II}}\, \mathrm{h}^q_{\mathrm{I}}\, \mathrm{F}^\bullet[0] & \Rightarrow & \mathrm{h}^n\, \mathrm{F}^\bullet \\ \downarrow & & \downarrow \\ \mathrm{h}^p_{\mathrm{II}}\, \mathrm{h}^q_{\mathrm{I}}\, \mathrm{G}^{\bullet,\bullet} & \Rightarrow & \mathrm{h}^n\, \operatorname{tot} \mathrm{G}^{\bullet,\bullet}. \end{array}$$

Both spectral sequences degenerate because of

$$\mathrm{h}^p_{\mathrm{II}}\, \mathrm{h}^q_{\mathrm{I}}\, \mathrm{F}^\bullet[0] = \begin{cases} \mathrm{h}^p\, \mathrm{F}^\bullet & \text{if } q = 0, \\ 0 & \text{else} \end{cases}$$

and

$$\mathrm{h}^p_{\mathrm{II}}\, \mathrm{h}^q_{\mathrm{I}}\, \mathrm{G}^{\bullet,\bullet} = \begin{cases} \mathrm{h}^p\, \mathrm{F}^\bullet & \text{if } q = 0, \\ 0 & \text{else} \end{cases}$$

and so we get an isomorphism on the initial terms. Hence we also have an isomorphism on the limit terms and our assertion follows. □

Remark 2.2.6. A similar statement holds with $\mathrm{F}^\bullet[0]$ considered as a double complex concentrated in the zeroth column.

Godement resolutions. As we also need Γ-acyclic resolutions that behave functorial in the proof of Corollary 2.2.4, we now describe the concept of Godement resolutions. For any sheaf $\mathcal{F}$ on X (or X^{an}) define

$$g(\mathcal{F}) := \prod_{x \in |X|} i_* \mathcal{F}_x,$$

where $i : x \hookrightarrow X$ denotes the closed immersion of the point x. The sheaf $g(\mathcal{F})$ is flabby and we have a natural inclusion

$$\mathcal{F} \rightarrowtail g(\mathcal{F}).$$

Setting $\mathcal{G}^{i+1} := g\left(\operatorname{coker}(\mathcal{G}^{i-1} \to \mathcal{G}^i)\right)$ with $\mathcal{G}^{-1} := \mathcal{F}$ and $\mathcal{G}^0 := g(\mathcal{F})$ gives an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{G}^1 \longrightarrow \dots .$$

We define the *Godement resolution* of $\mathcal{F}$ to be

$$\mathcal{G}^\bullet_{\mathcal{F}} := 0 \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{G}^1 \longrightarrow \dots .$$

It is Γ-acyclic and functorial in $\mathcal{F}$. Extending this definition to a bounded complex $\mathcal{F}^\bullet$

$$\begin{aligned} \mathcal{G}^{\bullet,q}_{\mathcal{F}^\bullet} &:= \mathcal{G}^\bullet_{\mathcal{F}^q} \\ \mathcal{G}^\bullet_{\mathcal{F}^\bullet} &:= \operatorname{tot} \mathcal{G}^{\bullet,\bullet}_{\mathcal{F}^\bullet} \end{aligned} \tag{6}$$

yields a map of double complexes

$$\mathcal{F}^\bullet[0] \longrightarrow \mathcal{G}^{\bullet,\bullet}$$

and a quasi-isomorphism by Lemma 2.2.5

$$\mathcal{F}^\bullet \xrightarrow{\sim} \mathcal{G}^\bullet_{\mathcal{F}^\bullet}.$$

Let $x \in |X|$ be a closed point of X. The map ϕ from (2) gives a commutative square

$$\begin{array}{ccc} \{x\}^{\mathrm{an}} & \stackrel{i}{\hookrightarrow} & X^{\mathrm{an}} \\ \phi \downarrow & & \downarrow \phi \\ \{x\} & \stackrel{i}{\hookrightarrow} & X \end{array}$$

and a natural morphism (cf. [Ha, II.5, p. 110])

$$\mathcal{F}_x \longrightarrow \phi_* \phi^* \mathcal{F}_x,$$

where we consider the stalk $\mathcal{F}_x$ as a sheaf on $\{x\}$. This map induces another map

$$i_* \mathcal{F}_x \longrightarrow i_* \phi_* \phi^* \mathcal{F}_x = (i \circ \phi)_* \phi^* \mathcal{F}_x = (\phi \circ i)_* \phi^* \mathcal{F}_x = \phi_* i_* \phi^* \mathcal{F}_x,$$

and yet a third map (cf. loc. cit.)

$$\varepsilon : \phi^* i_* \mathcal{F}_x \longrightarrow i_* \phi^* \mathcal{F}_x, \tag{7}$$

which is an isomorphism, as can be seen on the stalks

$$\begin{aligned} \varepsilon_x &: \phi^*\mathcal{F}_x \xrightarrow{\sim} \phi^*\mathcal{F}_x, \\ \varepsilon_y &: \quad 0 \longrightarrow 0, \qquad \text{for } y \neq x. \end{aligned}$$

Consequently,

$$\phi^* g(\mathcal{F}) = \phi^* \prod_x i_*\mathcal{F}_x = \prod_x \phi^* i_* \mathcal{F}_x \overset{(7)}{=} \prod_x i_*\phi^*\mathcal{F}_x = \prod_x i_*(\phi^*\mathcal{F})_x = g(\phi^*\mathcal{F}_x).$$

Thus we get first

$$\phi^*\mathcal{G}^\bullet_{\mathcal{F}^p} = \mathcal{G}^\bullet_{\phi^*\mathcal{F}^p},$$

then

$$\phi^*\mathcal{G}^\bullet_{\mathcal{F}^\bullet} = \mathcal{G}^\bullet_{\phi^*\mathcal{F}^\bullet}.$$

This gives us a natural map of hypercohomology groups

$$\begin{aligned} \mathbb{H}^i(X;\mathcal{F}^\bullet) = \mathrm{h}^i\Gamma(X;\mathcal{G}^\bullet_{\mathcal{F}^\bullet}) = \mathrm{h}^i\Gamma(X^{\mathrm{an}};\phi^{-1}\mathcal{G}^\bullet_{\mathcal{F}^\bullet}) \\ \longrightarrow \\ \mathrm{h}^i\Gamma(X^{\mathrm{an}};\phi^*\mathcal{G}^\bullet_{\mathcal{F}^\bullet}) = \mathrm{h}^i\Gamma(X^{\mathrm{an}};\mathcal{G}^\bullet_{\phi^*\mathcal{F}^\bullet}) = \mathrm{h}^i\Gamma(X^{\mathrm{an}};\mathcal{G}^\bullet_{\mathcal{F}^\bullet_{\mathrm{an}}}) = \mathbb{H}^i(X^{\mathrm{an}};\mathcal{F}^\bullet_{\mathrm{an}}). \quad (8) \end{aligned}$$

Proof of Corollary 2.2.4. We claim that the natural map (8) is an isomorphism for all i if X is projective.
If $\mathcal{F}^\bullet$ has has length one, Theorem 2.2.3 tells us that this is indeed true. So let us assume that (8) is an isomorphism for all complexes of length $\leq n$ and let $\mathcal{F}^\bullet$ be a complex of coherent sheaves on X of length $n+1$. W.l.o.g. $\mathcal{F}^{n+1} \neq 0$ but $\mathcal{F}^p = 0$ for $p > n+1$. We write $\sigma_{\leq n}\mathcal{F}^\bullet$ for the complex $\mathcal{F}^\bullet$ cut off above degree n. The short exact sequence

$$0 \longrightarrow \mathcal{F}^{n+1}[-n-1] \longrightarrow \mathcal{F}^\bullet \longrightarrow \sigma_{\leq n}\mathcal{F}^\bullet \longrightarrow 0$$

remains exact if we take the inverse image along ϕ

$$0 \longrightarrow \mathcal{F}^{n+1}_{\mathrm{an}}[-n-1] \longrightarrow \mathcal{F}^\bullet_{\mathrm{an}} \longrightarrow \sigma_{\leq n}\mathcal{F}^\bullet_{\mathrm{an}} \longrightarrow 0.$$

Using the naturality of (8), we obtain the following "ladder" with commuting squares

$$\begin{array}{ccccccc} \cdots \to \mathrm{H}^{-n-1+i}(X;\mathcal{F}^{n+1}) & \longrightarrow & \mathbb{H}^i(X;\mathcal{F}^\bullet) & \longrightarrow & \mathbb{H}^i(X;\sigma_{\leq n}\mathcal{F}^\bullet) \to \cdots \\ \downarrow\wr & & \downarrow & & \downarrow\wr \\ \cdots \to \mathrm{H}^{-n-1+i}(X^{\mathrm{an}};\mathcal{F}^{n+1}_{\mathrm{an}}) & \longrightarrow & \mathbb{H}^i(X^{\mathrm{an}};\mathcal{F}^\bullet_{\mathrm{an}}) & \longrightarrow & \mathbb{H}^i(X^{\mathrm{an}};\sigma_{\leq n}\mathcal{F}^\bullet_{\mathrm{an}}) \to \cdots \end{array}$$

and the induction step follows from the 5-lemma. □

3 Algebraic deRham Theory

In this section, we define the algebraic deRham cohomology $\mathrm{H}^\bullet_{\mathrm{dR}}(X, D/k)$ of a smooth variety X over a field k and a normal-crossings-divisor D on X (cf. definitions 3.2.3, 3.2.4, and 3.2.6). We also give some working tools for this cohomology: a base change theorem (Proposition 3.5.1) and two spectral sequences (Corollary 3.6.3 and Proposition 3.6.4).

3.1 Classical deRham Cohomology

This subsection only serves as a motivation for the following giving an overview of classical deRham theory. For a complex manifold, analytic deRham, complex, and singular cohomology are defined and shown to be equal to classical deRham cohomology. All material presented here will be generalized to a relative setup later on.

Let M be a complex manifold. We recall two standard exact sequences,

(i) the *analytic deRham complex* of holomorphic differential forms on M

$$0 \longrightarrow \mathbb{C}_M \longrightarrow \Omega^0_M \xrightarrow{\partial} \Omega^1_M \xrightarrow{\partial} \ldots, \text{ and}$$

(ii) the *classical deRham complex* of smooth $\mathbb{C}$-valued differential forms on M

$$0 \longrightarrow \mathbb{C}_M \longrightarrow \mathcal{E}^0_M \xrightarrow{d} \mathcal{E}^1_M \xrightarrow{d} \ldots,$$

where $\mathbb{C}_M$ is the constant sheaf with fibre $\mathbb{C}$ on M.
In both cases exactness is a consequence of the respective Poincaré lemmas [W, 4.18, p. 155] and [GH, p. 25].
Now consider the commutative diagram

$$\begin{array}{ccc} \mathbb{C}_M[0] & = & \mathbb{C}_M[0] \\ \downarrow\wr & & \downarrow\wr \\ \Omega^\bullet_M & \hookrightarrow & \mathcal{E}^\bullet_M. \end{array}$$

We can rephrase the exactness of the sequences (i) and (ii) by saying that the vertical maps are quasi-isomorphisms. We indicate quasi-isomorphisms by a tilde. Hence the natural inclusion

$$\Omega^\bullet_M \hookrightarrow \mathcal{E}^\bullet_M$$

is a quasi-isomorphism as well. Therefore the hypercohomology of the two complexes coincides

$$\mathbb{H}^\bullet(M; \Omega^\bullet_M) = \mathbb{H}^\bullet(M; \mathcal{E}^\bullet_M).$$

The sheaves $\mathcal{E}^p_M$ are fine, since they admit a partition of unity. In particular they are acyclic for the global section functor $\Gamma(M, ?)$ and we obtain

$$\mathbb{H}^\bullet(M; \mathcal{E}^\bullet_M) = \mathrm{h}^\bullet\Gamma(M; \mathcal{E}^\bullet_M).$$

The right-hand-side is usually called the *classical deRham cohomology* of M, denoted

$$\mathrm{H}^\bullet_{\mathrm{dR}}(M; \mathbb{C}).$$

The equalities above give

$$\mathrm{H}^{\bullet}_{\mathrm{dR}}(M;\mathbb{C}) = \mathbb{H}^{\bullet}(M;\Omega^{\bullet}_{M}).$$

We refer to the right-hand-side $\mathbb{H}^{\bullet}(M;\Omega^{\bullet}_{M})$ as *analytic deRham cohomology*, for which we want to use the same symbol $\mathrm{H}^{\bullet}_{\mathrm{dR}}(M;\mathbb{C})$. The hypercohomology $\mathbb{H}^{\bullet}(M;\Omega^{\bullet}_{M})$ turns out to be a good candidate for generalizing deRham theory to algebraic varieties. Both variants of deRham cohomology agree with *complex cohomology*

$$\mathrm{H}^{\bullet}(M;\mathbb{C}) := \mathrm{H}^{\bullet}(M;\mathbb{C}_M) = \mathbb{H}^{\bullet}(M;\mathbb{C}_M[0]).$$

We have yet a third resolution of the constant sheaf $\mathbb{C}_M$ given by the *complex of singular cochains*: For any open set $U \subseteq M$, we write $\mathrm{C}^{p}_{\mathrm{sing}}(U;\mathbb{C})$ for the vector space of singular p-cochains on U with coefficients in $\mathbb{C}$. The sheaf of singular p-cochains is now defined as

$$\mathcal{C}^{p}_{\mathrm{sing}}(M;\mathbb{C}) : U \mapsto \mathrm{C}^{p}_{\mathrm{sing}}(U;\mathbb{C}) \quad \text{for} \quad U \subseteq M \quad \text{open}$$

with the obvious restriction maps. The sheaves $\mathcal{C}^{p}_{\mathrm{sing}}(M;\mathbb{C})$ are flabby: The restriction maps are the duals of injections between vector spaces of singular p-chains and the functor $\mathrm{Hom}_{\mathbb{C}}(?,\mathbb{C})$ is exact. In particular these sheaves are acyclic for the global section functor $\Gamma(M;?)$ and we obtain

$$\mathbb{H}^{\bullet}(M;\mathcal{C}^{\bullet}_{\mathrm{sing}}(M;\mathbb{C})) = \mathrm{h}^{\bullet}\Gamma(M;\mathcal{C}^{\bullet}_{\mathrm{sing}}(M;\mathbb{C})) = \mathrm{h}^{\bullet}\mathrm{C}^{\bullet}_{\mathrm{sing}}(M;\mathbb{C}) = \mathrm{H}^{\bullet}_{\mathrm{sing}}(M;\mathbb{C}).$$

By the following lemma, we conclude $\mathrm{H}^{\bullet}(M;\mathbb{C}) = \mathrm{H}^{\bullet}_{\mathrm{sing}}(M;\mathbb{C})$.

Lemma 3.1.1. *For any locally contractible, locally path-connected topological space M the sequence*

$$0 \longrightarrow \mathbb{C}_M \longrightarrow \mathcal{C}^{0}_{\mathrm{sing}}(M;\mathbb{C}) \longrightarrow \mathcal{C}^{1}_{\mathrm{sing}}(M;\mathbb{C}) \longrightarrow \ \ldots\ .$$

is exact.

Proof. Note first that $\mathbb{C}_M = \mathrm{h}^0\,\mathcal{C}^{\bullet}_{\mathrm{sing}}(M;\mathbb{C})$, since M is locally path-connected. For the higher cohomology sheaves $\mathrm{h}^p\,\mathcal{C}^{\bullet}_{\mathrm{sing}}(M;\mathbb{C})$, $p > 0$, we observe that any element s_x of the stalk $\mathrm{h}^p\,\mathcal{C}^{\bullet}_{\mathrm{sing}}(M;\mathbb{C})_x$ at $x \in M$ not only lifts to a section s of $\mathrm{C}^{p}_{\mathrm{sing}}(U;\mathbb{C})$ for some contractible open subset $x \in U \subset M$, but that we can assume s to be a cocycle by eventually shrinking U. Now this s is also a coboundary because of

$$\mathrm{h}^p\mathrm{C}^{\bullet}_{\mathrm{sing}}(U;\mathbb{C}) = \mathrm{H}^{p}_{\mathrm{sing}}(U;\mathbb{C}) = 0 \quad \text{for all} \quad p > 0.$$

□

We summarize this subsection in the following proposition.

Proposition 3.1.2. *Let M be a complex manifold. Then we have a chain of natural isomorphisms between the various cohomology groups defined in this subsection*

$$\mathrm{H}^{\bullet}_{\mathrm{dR}}(M;\mathbb{C}) \cong \mathbb{H}^{\bullet}(M;\Omega^{\bullet}_{M}) \cong \mathrm{H}^{\bullet}(M;\mathbb{C}) \cong \mathrm{H}^{\bullet}_{\mathrm{sing}}(M;\mathbb{C}).$$

3.2 Algebraic deRham Cohomology

Let X be a smooth variety defined over a field k and D a divisor with normal crossings on X; where having normal crossings means, that locally D looks like a collection of coordinate hypersurfaces, or more precisely:

Definition 3.2.1 (Divisor with normal crossings, [Ha, p. 391]). *A divisor $D \subset X$ is said to have normal crossings, if each irreducible component of D is nonsingular and whenever s irreducible components $D_1, \ldots, D_s$ meet at a closed point P, then the local equations $f_1, \ldots, f_s$ of the D_i form part of a regular system of parameters $f_1, \ldots, f_d$ at P.*

It is proved in [Ma, 12, p. 78] that in this case the $f_1, \ldots, f_d$ are linearly independent modulo m_P^2, where m_P is the maximal ideal of the local ring $\mathcal{O}_{X,P}$ at P. By the inverse function theorem for holomorphic functions [Gun, Thm. 6, p. 20], we find in a neighbourhood of any $P \in X^{\mathrm{an}}$ a holomorphic chart $z_1, \ldots, z_d$ such that D^{an} is given as the zero-set $\{z_1 \cdot \ldots \cdot z_s = 0\}$.
We are now going to define algebraic deRham cohomology groups

$$\mathrm{H}^\bullet_{\mathrm{dR}}(X/k), \quad \mathrm{H}^\bullet_{\mathrm{dR}}(D/k) \quad \text{and} \quad \mathrm{H}^\bullet_{\mathrm{dR}}(X, D/k).$$

Remark 3.2.2. In [Ha:dR], algebraic deRham cohomology is de[illegible]rieties with arbitrary singularities. However, the relative version of algebr[illegible]ohomology discussed in [Ha:dR] deals with m[illegible]sms of varieties, [illegible]

3.2.1 The Smooth Case

Definition 3.2.3 (Algebraic deRham cohom[illegible]gy for a smo[illegible] *We set*

$$\mathrm{H}^\bullet_{\mathrm{dR}}(X/k) := \mathbb{H}^\bullet(X; \Omega^\bullet_{X/k}),$$

where $\Omega^\bullet_{X/k}$ is the complex of algebraic differential forms o[illegible] over k (cf. [Ha, II.8, p. 175]).

3.2.2 The Case of a Divisor with Normal Cros[illegible]