Asymptotic Behaviour of Capillary Problems governed by Disjoining Pressure Potentials

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Die Annahme der Dissertation wurde empfohlen von:

1. Prof. Dr. Erich Miersemann, Universität Leipzig

2. Prof. Dr. John McCuan, Georgia Institute of Technology

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Introduction

Capillarity describes the effects caused by the surface tension on liquids. When considering small amounts of liquid, the surface tension becomes the dominating parameter. In this situation the arising mathematical task is to determine the occurring capillary surface. At the beginning of the research on this topic, problems such as the ascent of fluids in a circular tube, on a vertical wall or on a wedge were some of the first problems scientists were concerned with. At the beginning of the 19th century, scientists like Young[1], Laplace[2], Taylor [3] and Gauss[4] established the mathematical foundations of this field. For the *capillary tube*[5] they found, by applying variational methods, the so called *mean curvature equation* or *capillary equation* with the associated boundary condition. As Finn in [Fin86, Chapter 1] describes, this leads to the following boundary value problem:

$$\begin{aligned} \operatorname{div} \mathbf{T}\, u &= \kappa u + \lambda && \text{in } \Omega, \\ \nu \cdot \mathbf{T}\, u &= \cos\gamma && \text{on } \partial\Omega, \end{aligned}$$

where $\mathbf{T}\, u = \dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}$. λ is called the Lagrange[6] multiplier and γ is the contact angle, established between the capillary surface and the container wall.

In the past, one tried to solve the problem by linearisation – with more or less satisfying results. In the last decades, expedited by the developing of micromechanics and the arising space-technology, capillary effects became more and more significant. Thereby the observed results differed from the predicted. The reason is the strong non-linearity of the problem.

Interior molecular forces are responsible for the establishing of equilibrium surfaces. The force, operating between two materials, is called adhesion and cohesion is the molecular force within a medium. Under some specifications there arises a non-negligible force, called disjoining pressure. This pressure causes an additional term in the capillary equation, which

[1]Thomas Young (*13 June 1773, Milverton; †10 May 1829, London); Englisch polymath; made notable contributions to the fields of vision, light, solid mechanics, energy, physiology, language, musical harmony and Egyptology, found the Young–Laplace equation

[2]Pierre-Simon (Marquis de) Laplace (28 March 1749, Beaumont-en-Auge; †5 March 1827, Paris); French mathematician and astronomer; found the Young–Laplace equation

[3]Brook Taylor (*18 August 1685, Edmonton; †29 December 1731, Somerset House/London); English mathematician; experiments in capillary attraction

[4]Johann Carl Friedlich Gauß (*30 April 1777, Braunschweig; †23 February 1855, Göttingen); German mathematician and scientist; contributed significantly to many fields, including number theory, statistics, analysis, differential geometry, geodesy, geophysics, electrostatics, astronomy and optics

[5]A capillary tube is a container with cross-section Ω and perpendicular container walls, which contains an amount of liquid.

[6]Joseph-Louis de Lagrange (*25 January 1736, Turin; †10 April 1813, Paris); Italian mathematician and astronomer.

is called the *disjoining pressure potential*, denoted by $P(x, u(x))$. That is, we are led to the following modified capillary equation, see [MMS08]:

$$\operatorname{div} \mathbf{T}\, u = \kappa u + P + \lambda \qquad \text{in } \Omega,$$

with a similar boundary condition (see Section 1.3 for more details). The main task of this paper is to examine the behaviour of the capillary problem, considering the disturbance P. A generic example for such configurations is *vapour nitrogen//liquid nitrogen//quartz*, see also [Isr92, Chapter 11] or [MMS08].

The present work with regard to contents is divided in three parts. In the first part, inspired by the work of Concus and Finn [CF74], [FH89], we prove a Comparison Principle. As in the classical context, this principle is a powerful tool to find solutions of the boundary problem. Thus we can see that the disjoining pressure potential is the key for the asymptotic of the solutions.

The second part is concerned with the asymptotic behaviour of the solutions for some classical cases. In particular for the capillary tube with circular cross-section (see [Mie93b], [Mie94], [Mie96] for the classical setting) the ascent on a horizontal wall and between two parallel horizontal plates, results are presented. There we are able to specify the asymptotic behaviour up to a constant term.

In the last part we observe the solution of the problem on a corner. There it is more difficult to obtain a result. But in return, we gain a better result near the cusp of the edge. In the articles of Miersemann [Mie88], [Mie89], [Mie90] or Scholz [Sch04] some results for the classical setting are given.

The formal arrangement is divided into three main chapters. The first of them is a summary of some notations which will be needed in the following chapters and also the physical background is illuminated. The main part, where asymptotic results are presented, is contained in Chapter 2. To afford a better reading, most of the proofs are given in Chapter 3.

Chapter 1

Notations and physical background

In this section we will outline some mathematical facts which will be required in the subsequent chapters. Since the divergence of a function plays a major role, we will have a closer look at it. It is also essential to perform some large calculations. In some of these calculations it is sufficient to estimate some terms. Thereby it is adjuvant to use both the "big O notation" and the "asymptotic notation", which we will also introduce in the following. Finally we will give a brief overview of the physical background of the underlying problem.

1.1 Divergence in polar coordinates

Based on the definition of the divergence we compute it by using polar coordinates. In the later sections this description enables us to obtain some explicit results.

Let be $\Omega \subset \mathbb{R}^2$, $x = (x_1, x_2) \in \Omega$ and $\mathbf{T}\, u = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$, where $\nabla u = (u_{x_1}, u_{x_2})$ denotes the gradient of u. So we have by definition, for a function $u = u(x)$

$$\operatorname{div} \mathbf{T}\, u = \frac{\partial}{\partial x_1}\left(\frac{u_{x_1}}{\sqrt{1+|\nabla u|^2}}\right) + \frac{\partial}{\partial x_2}\left(\frac{u_{x_2}}{\sqrt{1+|\nabla u|^2}}\right),$$

where $|\nabla u|^2 = u_{x_1}^2 + u_{x_2}^2$. Let be

$$\operatorname{div} \mathbf{T}\, u = f \qquad \text{in } \Omega$$

for a given function $f = f(x)$ and let be $v = v(x) \in \mathcal{C}_c^1(\Omega)$. Thereby $C_c^1(\Omega)$ denotes the space of all continuous differentiable functions with compact support in Ω. Then we get, using the definition of $\operatorname{div} \mathbf{T}\, u$ and integration by parts

$$\begin{aligned} \int_\Omega f v \mathrm{d}x = \int_\Omega \operatorname{div} \mathbf{T}\, u \cdot v \mathrm{d}x &= \int_\Omega \left\{ \frac{\partial}{\partial x_1}\left(\frac{u_{x_1}}{\sqrt{1+|\nabla u|^2}}\right) v + \frac{\partial}{\partial x_2}\left(\frac{u_{x_2}}{\sqrt{1+|\nabla u|^2}}\right) v \right\} \mathrm{d}x \\ &= -\int_\Omega \left\{ \frac{u_{x_1}}{\sqrt{1+|\nabla u|^2}} v_{x_1} + \frac{u_{x_2}}{\sqrt{1+|\nabla u|^2}} v_{x_2} \right\} \mathrm{d}x. \end{aligned} \tag{1.1}$$

At this point we introduce the polar coordinates that is, we choose $x_1 = r\cos\varphi$ and $x_2 = r\sin\varphi$. So we get

$$u(x_1, x_2) = u(r\cos\varphi, r\sin\varphi) = \bar{u}(r, \varphi)$$

and so

$$\bar{u}_r = u_{x_1} \cos\varphi + u_{x_2} \sin\varphi,$$
$$\bar{u}_\varphi = -u_{x_1} r \sin\varphi + u_{x_2} r \cos\varphi$$

and hence

$$\begin{aligned} u_{x_1} &= \bar{u}_r \cos\varphi - r^{-1}\bar{u}_\varphi \sin\varphi, \\ u_{x_2} &= \bar{u}_r \sin\varphi + r^{-1}\bar{u}_\varphi \cos\varphi. \end{aligned} \tag{1.2}$$

Thus we can compute

$$|\nabla u|^2 = u_{x_1}^2 + u_{x_2}^2 = \bar{u}_r^2 + r^{-2}\bar{u}_\varphi^2.$$

Inserting polar coordinates in (1.1) and considering $\mathrm{d}x = r\mathrm{d}r\mathrm{d}\varphi$ yields

$$\begin{aligned} \int_{\bar{\Omega}} \bar{f}\bar{v} r \mathrm{d}r\mathrm{d}\varphi &= -\int_{\bar{\Omega}} \frac{\bar{u}_r \bar{v}_r + r^{-2}\bar{u}_\varphi \bar{v}_\varphi}{\sqrt{1+|\nabla u|^2}} r\mathrm{d}r\mathrm{d}\varphi \\ &= \int_{\bar{\Omega}} \left[\left(\frac{r\bar{u}_r}{\sqrt{1+|\nabla u|^2}} \right)_r + \left(\frac{r^{-1}\bar{u}_\varphi}{\sqrt{1+|\nabla u|^2}} \right)_\varphi \right] \bar{v} \mathrm{d}r\mathrm{d}\varphi, \end{aligned} \tag{1.3}$$

where we again used integration by parts. Applying the fundamental lemma of calculus of variations to equation (1.3) we get

$$\operatorname{div} \mathbf{T}\, u = r^{-1} \left\{ \left(\frac{r\bar{u}_r}{\sqrt{1+|\nabla u|^2}} \right)_r + \left(\frac{r^{-1}\bar{u}_\varphi}{\sqrt{1+|\nabla u|^2}} \right)_\varphi \right\}.$$

This is the desired result.

With regard to later applications, we also compute the boundary condition in polar coordinates of a wedge shaped domain that is, we define for $0 < \alpha < \pi$

$$\Omega = \left\{ (r, \theta) \in \mathbb{R}^2 : -\alpha \le \theta \le \alpha,\ r > 0 \right\},$$

see Figure 1.1. We have by (1.2)

$$\mathbf{T}\, u = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \frac{1}{\sqrt{1+|\nabla u|^2}} \begin{pmatrix} \bar{u}_r \cos\varphi - r^{-1}\bar{u}_\varphi \sin\varphi \\ \bar{u}_r \sin\varphi + r^{-1}\bar{u}_\varphi \cos\varphi \end{pmatrix}$$

and on Σ^+ that is, if $\theta = +\alpha$ and $r > 0$

$$\nu = \begin{pmatrix} \cos(\alpha + \pi/2) \\ \sin(\alpha + \pi/2) \end{pmatrix} = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix},$$

where ν is the outer normal. And so we get on Σ^+

$$\nu \cdot \mathbf{T}\, u = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} \bar{u}_r \cos\varphi - r^{-1}\bar{u}_\varphi \sin\varphi \\ \bar{u}_r \sin\varphi + r^{-1}\bar{u}_\varphi \cos\varphi \end{pmatrix} = \frac{1}{\sqrt{1+|\nabla u|^2}}.$$

By an similar calculation on Σ^-, we get the final result on $\Sigma \setminus \{0\}$

$$\nu \cdot \mathbf{T}\, u = \operatorname{sign}\theta \frac{1}{\sqrt{1+|\nabla u|^2}}, \qquad \theta = \pm\alpha.$$

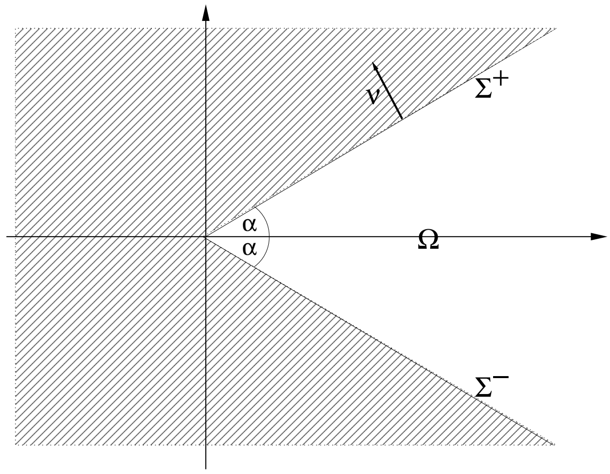

Figure 1.1: Wedge shaped domain.

1.2 Order notation and asymptotic notation

The "order notation"[1] and the "asymptotic notation" will be useful to simplify some calculations which makes them easier to understand. Thereto we use the definitions of Murray, see [Mur84, p. 2ff].

If $f(z)$ and $g(z)$, two functions of a complex number z, defined on some domain Ω, possess limits as $z \to z_0$ in Ω, then we say $f(z) = O(g(z))$ if there exist positive constants K and k such that $|f| \leq K|g|$ whenever $0 < |z - z_0| < k$. If $|f| \leq K|g|$ for all z in Ω, we say $f(z) = O(g(z))$ in Ω.

Now we specify the so called "order notation". Let Ω be a subset of $\mathbb{C}^n$, $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, and $f(z)$ and $g(z)$ defined on Ω. Then we introduce the 'Landau-notation' as follows.

Definition 1

1. $f(z) = O(g(z)) :\Leftrightarrow \exists A > 0 : |f| \leq A|g| \ \forall z \in \Omega$
2. $f(z) = O(g(z)),$ *as* $z \to z_0 :\Leftrightarrow \exists A, \delta > 0 : |f| \leq A|g| \ \forall z \in B_\delta(z_0) \cap \Omega$
3. $f(z) = o(g(z)),$ *as* $z \to z_0 :\Leftrightarrow \forall \varepsilon > 0 \ \exists \delta > 0 : |f| \leq \varepsilon|g| \ \forall z \in B_\delta(z_0) \cap \Omega$

For example, we have $\sin x = O(1)$ in $\mathbb{R}$, $\sin z = O(z)$ as $z \to 0$ and $\sin z = o(1)$ as $z \to 0$.

If $g \neq 0$ in Ω and in a neighbourhood of z_0 respectively we have got an equivalent formulation of the upper definition which is

$$f(z) = O(g(z)),\ \forall z \in \Omega \Leftrightarrow \left|\frac{f(z)}{g(z)}\right| \leq A < \infty,\ \forall z \in \Omega,$$

$$f(z) = O(g(z)),\ \text{as } z \to z_0 \Leftrightarrow \left|\frac{f(z)}{g(z)}\right| \leq A < \infty,\ \text{as } z \to z_0,$$

$$f(z) = o(g(z)),\ \text{as } z \to z_0 \Leftrightarrow \left|\frac{f(z)}{g(z)}\right| \to 0,\ \text{as } z \to z_0.$$

[1]The "order notation" is also known as "big O notation" or "Landau notion".

As mentioned above, if for a complicated function their asymptotic behaviour is known, it is together with the "order notation" an useful tool to shorten difficult calculations.

Definition 2
We say that a function $f(z)$ is ***asymptotically equal*** *to $g(z)$ under the limit $z \to z_0$ if f and g are such that $\lim\limits_{z \to z_0} \frac{f(z)}{g(z)} = 1$. Therefore we write also*

$$f(z) \sim g(z), \qquad \text{as } z \to z_0.$$

Sometimes, when mix-ups are impossible, we omit the argument of the function. For example, if $f(z) = z^{19} + z^7 \log z$, then $f \sim z^{19}$ as $z \to \infty$ and $f \sim z^7 \log z$ as $z \to 0$.

1.3 Physical background

At first we will sketch a short abstract about the practical use of the results presented in Chapter 2. Then we will give a survey of the underlying physical basics which lead to the difference to the classical capillary problem. Finally we dwell on some specific examples of *vapour//liquid//solid* configurations, on which our results can be applied.

Porous matter contains a filigree network of pores which are intricate and affiliated into each other[2]. If these porous materials get in contact with a liquid, capillary effects lead to an adsorption of the liquid. A major task is to determine the absorbed amount of substance in the pores. Therefore it is necessary to have statistics about the distribution of the shape and size of the occurring cavities. Each category of pores can absorb a fixed condensate quantity. To get the desired result for a specific type of pore, one examines in each case a single pore[3]. Thus combining the statistics of the cavities with the specific condensate quantity yields the total quantity of the absorbed liquid.

It is well known that all matter is built-on atoms and molecules. An atom itself consists of a nucleus (a conglomerate of protons and neutrons) and an atomic shell. In the shell electrons orbit the nucleus. It is also commonly known that electrons carry a negative charge and protons a positive one, while neutrons are uncharged, and atoms as a whole are neutral, too. The reason for the cohesion of atoms are the *strong interaction* and the *electrostatic force*, which are *short-range forces.*

But we are interested in the physical forces between atoms and molecules, in particular the so called *van der Waals* [4] *forces.* These forces are weaker compared to normal chemical bonds, but they play a fundamental role in physical chemistry, since they are *long range forces.* In the following we use the descriptions of Safran [Saf03, Chapter 5] and Israelachvili [Isr92, Chapters 3, 6] respectively.

Most of the types of physical forces arise from straightforward electrostatic interactions involving charged or bipolar molecules. But the van der Waals forces act between *all* atoms

[2]Unless otherwise noted, all nontrivial physical and chemical declarations arise from the correspondence with Peter Schiller, co-author of [MMS08].

[3]Results for single pores of some simple geometries are stated in [MMS08].

[4]Johannes Diderik van der Waals (*23 November 1837, Leiden; †8 March 1923, Amsterdam); Dutch physicist and thermodynamicist; important contributions on an equation of state for gases and liquids; 1910 Nobel Prize in Physics

and molecules even totally neutral ones. The dominating portions of the van der Waals forces are called *dispersion forces*. They play a major role in a host of important phenomena such as adhesion, surface tension, physical absorption, wetting, properties of liquids and thin films, and the strengths of solids, see [Isr92, 6.1]. Israelachvili summarised their main features as follows[5]:

(1) They are long-range forces and, depending on the situation, can be effective from large distances (greater than 10 nm) down to interatomic spacings (about 0.2 nm).
(2) These forces may be repulsive or attractive, and in general the dispersion force between two molecules or large particles does not follow a simple power law.
(3) Dispersion forces not only bring molecules together but also tend to mutually align or orient them, though this orienting effect is usually weak.
(4) The dispersion interaction of two bodies is affected by the presence of other bodies nearby. This is known as the *non-additivity* of an interaction.

In the following we are interested in the forces interacting between nonpolar atoms. So we will give a short heuristic derivation of some interesting physical phenomena.

As Israelachvili writes in [Isr92, 6.1], we can explain the arising dispersion forces as follows: The average duration of the dipole[6] of an atom is zero. But at any instant there exists a finite dipole moment, given by the instantaneous positions of the electrons about the nuclear protons. This dipole generates an electric field that polarises nearby atoms and inducing a dipole moment in them. The resulting interaction between this dipoles gives rise to an instantaneous attractive force between the atoms. That is, an instantaneous little variation of the electric characteristics of a single atom leads to an influence on the adjoining atoms.

As mentioned previously, this is a very simple illustration, primarily just valid for two molecules (or atoms) in a vacuum. In this case we always get an attractive force between the particles. Qualitatively one has the following dispersion interaction

$$u(r) = -\frac{\alpha_{ij}}{r^6}, \tag{1.4}$$

where r is the distance between the centre of the molecules and $\alpha_{ij} > 0$. Modelling a pore, three different media arise, that is, we have the following configuration

$$\text{diluted gas//condensed liquid film//solid body.} \tag{1.5}$$

So the dielectric properties of the interacting media influence the setting, too.

Let us consider two particles, a vapour one and a solid one, each of them near a boundary layer of the upper setting (1.5). To get the *effective interaction potential*, U, one has to add all interactions of the type, given in equation (1.4). In addition thereto, one also has to comprise the *solid-film-dispersions* and the *film-vapour-dispersions*. So Safran ([Saf03, Section 5.3]) describes that we are now led to the following formula of the effective interaction of the two particles

$$U = -\frac{\mathcal{H}}{\pi^2 \rho_v \rho_s} \frac{1}{r^6}.$$

Thereby ρ_v and ρ_s denote the particle-number-density[7] of vapour and solid respectively and r

[5] The following properties are taken from [Isr92, p. 83f].

[6] An electric dipole is a separation of positive and negative charge.

[7] That is, the number of particles per volume.

the distance between the center of the two particles. In this formula the *Hamaker*[8] *constant*, $\mathcal{H}$, occurs. The Hamaker constant depends on the *dielectric constants*[9] ε_v, ε_l and ε_s of the vapour, liquid and solid respectively. Simplified it can be computed by

$$\mathcal{H} = c \cdot (\varepsilon_v - \varepsilon_l)(\varepsilon_s - \varepsilon_l), \tag{1.6}$$

whereby c is a positive constant. From expression (1.6) we conclude that the *sign* of $\mathcal{H}$ depends on the differences between the dielectric constants. That is, the occurring dispersion interaction can be *attractive* or *repulsive* and the Hamaker constant specifies this property. More precisely if and only if the value of the dielectric constant of the liquid, ε_l, lies between ε_v and ε_s, $\mathcal{H}$ becomes negative and the van der Waals interaction is repulsive.

Summing up, the van der Waals force between two arbitrary molecules is attractive, but the overlapping of forces in a sequence of three phases can result in a repulsive force between the outer media. In this work we just consider the case of **repulsive van der Waals forces**. Repulsive means vividly that the *vapour//liquid* boundary layer and the *liquid//solid* boundary layer repulse each other.

The result of this repulsion is that the absorbed film will act to thicken the film, to lower its energy. But when the liquid climbs up the solid wall, the gain in van der Waals energy is at the expense of gravitational energy. So the equilibrium film thickness will decrease with the height, see Figure 1.2.

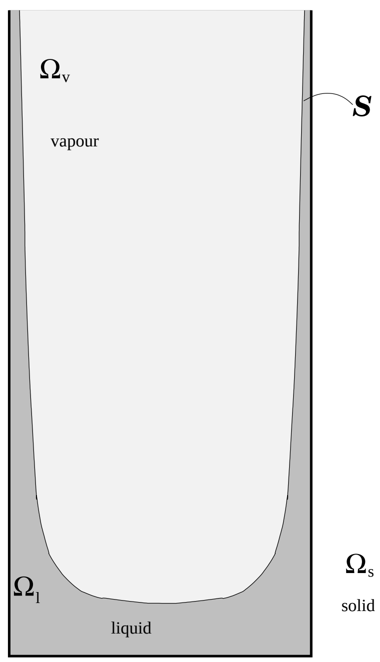

Figure 1.2: Ascent on the boundary.

[8]Hugo Christiaan Hamaker (*23 March 1905, Broek op Langendijk; †7 September 1993, Eindhoven); Dutch scientist; responsible for the Hamaker theory which explains the van der Waals forces between objects larger than molecules

[9]The dielectric constant is a measure of the extent of reduction of electric fields and consequently of the reduced stength of electrostatic interactions in a medium.

At this point ISRAELACHVILI introduces the *disjoining pressure* of a film, see [Isr92, 11.6], which arises when an equilibrium is established in the film. Just this disjoining pressure makes the difference to the classical capillary equation. Now, when $\mathcal{H} < 0$, the disjoining pressure has to be attended when the equilibrium of the three phase system is established. And after using a variational ansatz similar to the one in [Fin86, 1.7], we obtain the following boundary value problem

$$\begin{aligned} \operatorname{div} \mathbf{T}\, u &= \kappa u + P(x,u) + \lambda && \text{in } \Omega, \\ \nu \cdot \mathbf{T}\, u &= 1 && \text{on } \Sigma. \end{aligned} \tag{1.7}$$

The occurring expression $P(x,u)$ is the *disjoining pressure potential*, arising from the disjoining pressure. In general, the disjoining pressure potential of absorbed films on even and curved interfaces can be computed by

$$P = P(x,u(x)) = c \int_{\Omega_s} \left[(x_1 - y_1)^2 + (x_2 - y_2)^2 + (u - y_3)^2 \right]^{-p/2} \mathrm{d}y, \qquad p > 3. \tag{1.8}$$

Thereby Ω_s denotes the solid domain, see Figure 1.3. The occurring constants in the upper equations are given by the relations

$$\kappa = \rho g \sigma^{-1}, \qquad \lambda = \rho k T \ln(X) \sigma^{-1}, \qquad c = \mathcal{H} \pi^{-2} \sigma^{-1},$$

where

ρ is the difference between the number densities of the liquid and the vapour phase,

g is the gravitational constant (positive when the field is directed downward),

σ is the surface tension,

k is the Boltzmann constant,

T is the absolute temperature,

X is the reduced (constant) vapour pressure, $0 < X < 1$,

$\mathcal{H}$ is the (negative) Hamaker constant.

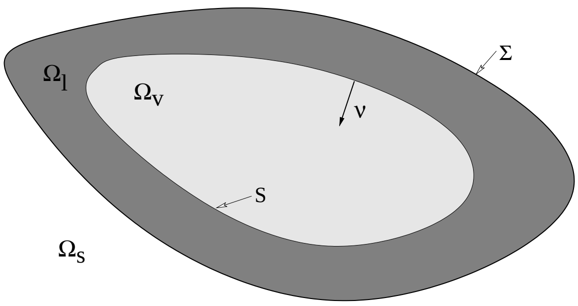

Figure 1.3: Solid, liquid and vapour domain.

In most practical cases the vapour density is negligibly small, so that ρ can be replaced by the density of the liquid phase. Since we consider the problem (1.7) for negative Hamaker constant $\mathcal{H}$ the occurring constant c is also *negative*.

As mentioned above, formula (1.8) is given in a general form. In the relevant cases we have $p = 6$. Cases with $3 < p < 6$ may arise in small systems without practical importance.

The most popular example is the configuration *helium vapour//liquid helium//solid*. Liquid helium avidly spreads on almost any surface. Thus if liquid helium is placed in a container, it rapidly climbs up the walls and down the other side. Eventually it leaves the beaker altogether. This behaviour results from the low dielectric constant, which is $\varepsilon_{\mathrm{He}} = 1.055$. And as mentioned above, the Hamaker constant becomes negative and the repulsive van der Waals forces act to thicken the film to lower its energy. This explains the strong ascent on the walls.

But since the extraction of helium is very expensive it is rarely used in adsorption-experiments. For financial reasons, one normally uses gases like nitrogen in such experiments[10].

Table 1.1: Static dielectric constants ε of some common liquids and solids at 25°C.

Compound		ε	Compound		ε
Hydrogen bonding			*Glasses*		
Methyl formamide	$HCONHCH_3$	182.4	Fused Quartz	SiO_2	3.8
Water	H_2O	78.5	Soda glass		7.0
Methanol	CH_3OH	32.6	Borosilicate glass		4.5
Ethanol	C_2H_5OH	24.3			
Ammonia	NH_3	16.9	*Crystalline solids*		
			Diamond (carbon)		5.7
Non-hydrogen bonding			Quartz	SiO_2	4.5
Acetone	$(CH_3)_2CO$	20.7	Sodium chloride	NaCl	6.0
Chloroform	$CHCl_3$	4.8			
Benzene	C_6H_6	2.3	*Miscellaneous*		
Carbon tetrachloride	CCl_4	2.2	Paraffin (liquid)		2.2
Hexane	C_6H_{14}	1.9	Paraffin wax (solid)		2.2
			Silicone oil		2.8
Polymers			Liquid helium (2-3K)		1.055
Nylon		3.7-4.2	Water (liquid, 0°C)		87.9
PTFE		2.0	Air (dry)		1.00054

The values of Table 1.1 are taken from [Isr92, 3.8].

In Table 1.1 the dielectric constants of some materials are given. Since hydrogen bonding is polar bonding, substances with hydrogen bonding exhibit the highest dielectric constants. We can also see that helium has a very low dielectric permittivity compared to other media. Just (dry) air and vacuum ($\varepsilon = 1$) have a lower permittivity. Dry air consists of nonpolar nitrogen and oxygen molecules. But still a very low air humidity falsifies the results and has therefore no practical importance for the upper explanations. Hence in adsorption-experiments one have to employ pure gases, for example one uses quartz in a pure nitrogen, oxygen or sometimes rare gases[11] atmosphere.

[10] For example determining the disturbation of the shape and size of pores

[11] Nitrogen, oxygen and rare gases are nonpolar.

Chapter 2

Asymptotic results

In this chapter the main results are given. But before we look at them, we will first repeat a well known result of CONCUS and FINN, namely the Comparison Principle for unbounded domains. Then we will specify a modified Comparison Principle, a resulting bound for the solution of problem (2.2), and in the last sections, we will give the asymptotic results for some common domains.

2.1 Normal Comparison Principle

We will state a classical version of the Comparison Principle of CONCUS and FINN, which was first published in [FH89][1]. The goal of this theorem is to estimate the behaviour of capillary surfaces in capillary tubes of general cross-section Ω. That is, we will search a graph u over a base domain $\Omega \subset \mathbb{R}^2$, with $\Sigma = \partial\Omega$, which suffices the boundary value problem

$$\begin{aligned} \operatorname{div} \mathbf{T}\, u &= \kappa u + \lambda && \text{in } \Omega, \\ \nu \cdot \mathbf{T}\, u &= \cos\gamma && \text{on } \Sigma, \end{aligned} \tag{2.1}$$

see [Fin86, Chapter 1] for details. In addition, the principle bears a lot of other applications. Some of them are listed in [Fin86, 5.2].

Actually the Comparison Principle is also valid for bounded and unbounded domains Ω in $\mathbb{R}^n$. But here just the statement for the case $n = 2$ is given, since this is physically reasonable. It is also worth noting that there is no growth hypothesis on u and ∇u in the Comparison Principle. This characteristic results from the particular nonlinearity of the equation.

Theorem 3 (Normal Comparison Principle)
Let $\kappa > 0$ and suppose $\Sigma = \partial\Omega$ admits a decomposition $\Sigma = \Sigma_\alpha \cup \Sigma_\beta \cup \Sigma_0$, where $\Sigma_\beta \in \mathcal{C}^1$ and Σ_0 has one-dimensional Hausdorff measure zero. Let $v, w \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\Sigma_\beta \cup \Omega)$ with the properties

(i) $\operatorname{div} \boldsymbol{T} w - \kappa w \geq \operatorname{div} \boldsymbol{T} v - \kappa v$ *in* Ω,
(ii) $w \leq v$ *as* $x \to \Sigma_\alpha$,
(iii) $\nu \cdot \boldsymbol{T} w \leq \nu \cdot \boldsymbol{T} v$ *as* $x \to \Sigma_\beta$.

Then we have $w \leq v$ in Ω.

By writing

$$\nu \cdot \mathbf{T}\, u \leq \nu \cdot \mathbf{T}\, v, \qquad \text{as } x \to \Sigma_\beta$$

[1] Versions for bounded domains have been published before.

we mean

$$\lim_{\substack{x \in \Omega \\ x \to x_0 \\ x_0 \in \Sigma_\beta}} \nu \cdot \mathbf{T}\, u \leq \lim_{\substack{x \in \Omega \\ x \to x_0 \\ x_0 \in \Sigma_\beta}} \nu \cdot \mathbf{T}\, v.$$

In property (ii) and in the following we use the corresponding abbreviation.

A simple consequence of the upper theorem is that the solution of the problem is unique and another one is the following corollary.

Corollary 4

Let $\kappa > 0$ and suppose $\Sigma = \partial\Omega$ admits a decomposition $\Sigma = \Sigma_\alpha \cup \Sigma_\beta \cup \Sigma_0$, where $\Sigma_\beta \in \mathcal{C}^1$ and Σ_0 has one-dimensional Hausdorff measure zero. Let v be a solution of (2.1), $w^\pm \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\Sigma_\beta \cup \Omega)$ with the properties

a) If

$$\begin{array}{rrcll} \text{(i)} & \operatorname{div} \boldsymbol{T} w^- - \kappa w^- & \geq & 0 & \textit{in } \Omega \\ \text{(i)} & w^- & \leq & v & \textit{as } x \to \Sigma_\alpha \\ \text{(iii)} & \nu \cdot \boldsymbol{T} w^- & \leq & \cos\gamma & \textit{as } \Sigma_\beta, \end{array}$$

then we have $w^- \leq v$ in Ω.

b) If

$$\begin{array}{rrcll} \text{(i)} & \operatorname{div} \boldsymbol{T} w^+ - \kappa w^+ & \leq & 0 & \textit{in } \Omega \\ \text{(i)} & w^+ & \geq & v & \textit{as } x \to \Sigma_\alpha \\ \text{(iii)} & \nu \cdot \boldsymbol{T} w^+ & \geq & \cos\gamma & \textit{as } \Sigma_\beta, \end{array}$$

then we have $w^+ \geq v$ in Ω.

This corollary bears the common recipe to find a solution of the upper boundary value problem (2.1). The strategy is to find suitable comparison functions $w^\pm$ which satisfy the corresponding properties (i)–(iii) of Corollary 4. Normally the comparison functions should have the shape $w^\pm(x) = f(x) \pm g(x)$ for some suitable functions f and g. Then Corollary 4 yields

$$w^- \leq v \leq w^+ \text{ in } \Omega,$$

that is, if $w^\pm$ has got the above mentioned shape, one gets

$$|v(x) - f(x)| \leq g(x) \text{ in } \Omega.$$

The best case is that the function g can be chosen arbitrarily small. Normally g can not be chosen small. In this case at least an estimate of the solution or an asymptotic expansion may be achieved by applying the Comparison Principle successively.

Another question is how to find a suitable comparison function. There is no general recipe, but often the underlying geometry of the problem provides an adapted function.

2.2 Modified Comparison Principle

A lot of research has been done concerning the classical capillary problem (2.1) for the capillary tube. In this section we will examine this problem with a perturbation, which is the disjoining pressure potential $P(x,u)$ which was mentioned previously. That is, we have the modified boundary value problem

$$\begin{aligned} \operatorname{div} \mathbf{T} u &= \kappa u + P(x,u) + \lambda && \text{in } \Omega, \\ \nu \cdot \mathbf{T} u &= 1 && \text{as } x \to \Sigma, \end{aligned} \tag{2.2}$$

for $\kappa > 0$, whereby P is defined in Section 1.3.

The main tool to study this problem is a modification of the Comparison Principle of Concus and Finn, which we formulate in the following theorem.

Theorem 5 (Modified Comparison Principle)
Let $\kappa > 0$ and suppose $\Sigma = \partial\Omega$ admits a decomposition $\Sigma = \Sigma_\alpha \cup \Sigma_\beta \cup \Sigma_0$, where $\Sigma_\beta \in \mathcal{C}^1$ and Σ_0 has one-dimensional Hausdorff measure zero. Let $u, v \in \mathcal{C}^2(\Omega)$ with the properties

(i) $\operatorname{div} \boldsymbol{T} u - \kappa u - P(x,u) \geq \operatorname{div} \boldsymbol{T} v - \kappa v - P(x,v)$ *in* Ω,

(ii) $u \leq v$ *as* $x \to \Sigma_\alpha$,

(iii) $\nu \cdot \boldsymbol{T} u \leq \nu \cdot \boldsymbol{T} v$ *as* $x \to \Sigma_\beta$.

Then we have $u \leq v$ in Ω.

The result will be that the case $P(x,u)$ is independent of the special graph u, describes a common situation. In the following we will have a closer look on this situation. If in addition $\operatorname{div} \mathbf{T} P$ is uniformly bounded, we can infer the asymptotic behaviour of the solution of problem (2.2) from the upper Comparison Principle:

Let v be a solution of the boundary problem (2.2), with $\Sigma_\alpha = \emptyset$ and $P(x,u)$ independent of the particular graph u. So let us denote it again by $P(x)$. And in addition let $\operatorname{div} \mathbf{T} P$ be bounded. With the notations of Theorem 5, the solution v satisfies in particular

$$\operatorname{div} \mathbf{T} v - \kappa v = P(x) - \lambda \qquad \text{in } \Omega \tag{2.3}$$

and

$$\nu \cdot \mathbf{T} v \to 1, \qquad \text{as } x \to \Sigma_\beta.$$

Now define $w_0 = \kappa^{-1}(\lambda - P(x))$ and $w^+ = w_0 + A$, for a positive $A \in \mathbb{R}$, to get the estimate

$$\begin{aligned} \operatorname{div} \mathbf{T} w^+ - \kappa w^+ &= \operatorname{div} \mathbf{T} w_0 - \kappa w_0 - \kappa A \\ &= \operatorname{div} \mathbf{T} w_0 - \kappa A + P(x) - \lambda \\ &\overset{(2.3)}{=} \operatorname{div} \mathbf{T} w_0 - \kappa A + \operatorname{div} \mathbf{T} v - \kappa v \\ &\leq \operatorname{div} \mathbf{T} v - \kappa v, \end{aligned}$$

for a constant A which satisfies $\operatorname{div} \mathbf{T} w_0 - \kappa A \leq 0$. Such a constant exists, since $\kappa > 0$ and $\operatorname{div} \mathbf{T} w_0$ is bounded by assumption. We have by definition of P

$$|P(x)| \to \infty, \qquad \text{as } x \to \Sigma_\beta$$

and so

$$\nu \cdot \mathbf{T} w^+ \to 1, \qquad \text{as } x \to \Sigma_\beta.$$

That is, in general w^+ satisfies the boundary condition (iii) of the Comparison Principle. So we can apply the principle to get $v \leq w^+$.

An analogous result arises for $w^- = w_0 - A$, so that we get the following theorem.

Theorem 6
Let $\kappa > 0$ and suppose $\Sigma = \partial\Omega$ admits a decomposition $\Sigma = \Sigma_\beta \cup \Sigma_0$, where $\Sigma_\beta \in \mathcal{C}^1$ and Σ_0 has one-dimensional Hausdorff measure zero and let the following properties be satisfied.
(i) *$P(x, u)$ is independent of $u(x)$*
(ii) *$\operatorname{div} \boldsymbol{T} P$ is uniformly bounded in Ω*
Then there is a constant $A \geq 0$, such that for the solution $v(x)$ of equation (2.2) follows

$$\left|v(x) + \kappa^{-1}P(x)\right| \leq A \qquad \text{in } \Omega.$$

Thus, Theorem 6 provides the asymptotic behaviour of the solution of problem (2.2). That is, $v \sim -\kappa^{-1}P(x)$, as $x \to \partial\Omega$.

Therefore in the modified problem the disjoining pressure potential P affords access to the solution of problem (2.2) – at first, just in the upper mentioned special case. But it will turn out that it also can be a powerful tool if $\operatorname{div} \mathbf{T} P$ is unbounded (see Section 2.6, for example).

2.3 Boundedness of v at the inner points

We will now present a method to obtain the boundedness of the solution v of (2.2) in a compact subset of an arbitrary domain Ω. Thereby we use an idea of Finn [Fin86, Theorem 5.2] and adapt it to the current circumstances.

The following lemma allows us to verify condition (ii) of Theorem 5 in some cases.

Lemma 7
Let be $\Omega \subset \mathbb{R}^2$ and let v define a capillary surface over the domain Ω, so that (2.2) holds with $\kappa > 0$.
Then v is bounded on every compact subset of Ω.

2.4 Capillary tube

Now we consider the special case that the cylinder has got a circular cross-section with constant radius $R > 0$. We can choose the domain as $\Omega = B_R(0)$, where $B_R(0) = \left\{x_1^2 + x_2^2 < R^2\right\}$. Thus we are looking for a rotationally symmetric solution, depending only on the distance from the origin, which is $r = \sqrt{x_1^2 + x_2^2}$. The scaling $q = r/R$ transforms the domain into $B_1(0)$.

At first we assume the case of an infinite tube, see Figure 2.1 and Figure 2.2. In this case we obtain an explicit result via integration. Here the solid domain is

$$\Omega_s = \left\{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : \; y_1^2 + y_2^2 > 1, \; -\infty < y_3 < \infty\right\}.$$

And so we get the following theorem.

Theorem 8
Let v be the solution of (2.2) over the domain $B_1(0)$. Then there is a constant $A \geq 0$, with

$$\left|v(q) - C \cdot \mathrm{F}(a, b; 1; q^2)\right| \leq A, \qquad \text{in } B_1(0),$$

where $a = \frac{p-3}{2}$, $b = \frac{p-1}{2}$, $C = -\pi^{-\frac{1}{2}}\mathcal{H}\kappa^{-1}\sigma^{-1}R^{3-p}\Gamma(a)/\Gamma(p/2) > 0$ and $\mathrm{F}(a, b; 1; q^2)$ is the hypergeometric Function.

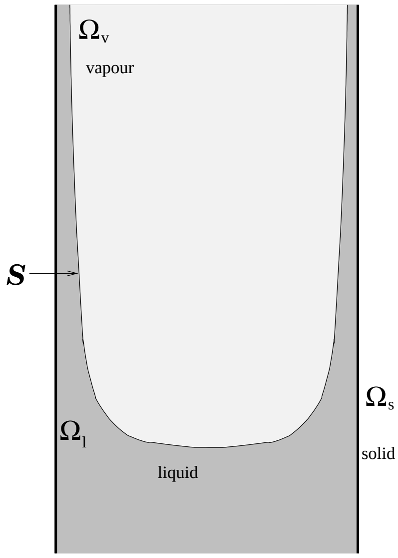

Figure 2.1: Open capillary tube.

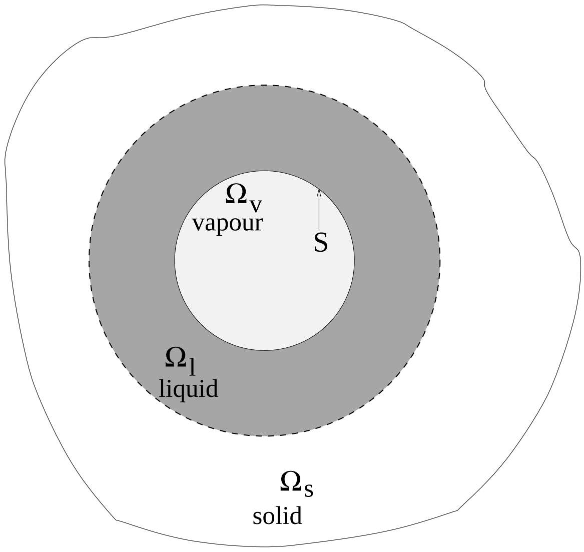

Figure 2.2: View from above.

For the **closed capillary tube** when we have the solid domain

$$\Omega_s = \mathbb{R}^3 \setminus \{y \in \mathbb{R}^3 : y_1^2 + y_2^2 \leq 1,\ y_3 \geq 0\},$$

one gets the same result, if one claims that the bottom of the tube is covered by the liquid. In other words, when $v(0) > 0$ is assumed.

In addition we present an explicit example for the physically significant case $p = 6$. Again let v be the solution of (2.2), then we get with [AS64, 15.3.12] (and $m = 3$) and the function

$$\tilde{v}(q) = \tilde{C} \cdot \left\{ \left(1 - q^2\right)^{-3} - \frac{3}{8}\left(1 - q^2\right)^{-2} - \frac{3}{32}\left(1 - q^2\right)^{-1} \right\}, \tag{2.4}$$

whereby $\tilde{C} = -\dfrac{4\mathcal{H}}{3\kappa\sigma\pi R^3} > 0$, the explicit result:

$$|v(q) - \tilde{v}(q)| \leq A \qquad \text{in } B_1(0), \tag{2.5}$$

with some positive constant A. From formulae (2.4) and (2.5) we see that v has got a leading singularity $\tilde{C}(1 - q^2)^{-3}$ and some lower singularities. The leading term is the same Israelachvili conjectured in 1985, see [Isr92, p. 194].

2.5 Vertical wall and parallel plates

We will now start to examine the ascent of a fluid on an infinite vertical wall. We choose the coordinate system in a way that the wall coincides with the $x_2 - x_3-$plane, see Figure 2.3.

In this situation we are looking for a solution depending just on the distance to the x_2-x_3-plane which is x_1. Here the domain Ω coincides with the half-plane $\{x \in \mathbb{R}^2 : x_1 \geq 0,\ -\infty < x_2 < \infty\}$ and we have $\Sigma = \{x_1 = 0,\ -\infty < x_2 < \infty\}$. Then we get the following result.

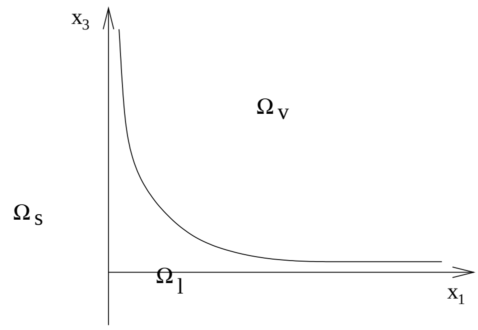

Figure 2.3: Ascent on a wall.

Theorem 9
Let v be the solution of the problem (2.2) over the domain $\Omega = \{x_1 > 0, -\infty < x_2 < \infty\}$. Then there is a constant $A \geq 0$ with

$$\left| v(x_1) - C \cdot x_1^{3-p} \right| \leq A \qquad \text{in } \Omega,$$

where $C = \dfrac{-2\mathcal{H}}{\pi\kappa\sigma(p-2)(p-3)} > 0$.

That is, for $p = 6$ we have in particular $C = -\dfrac{\mathcal{H}}{6\pi\kappa\sigma} > 0$ and so

$$v \sim -\frac{\mathcal{H}}{6\pi\kappa\sigma} x_1^{-3}, \qquad \text{as } x_1 \to 0.$$

A similar situation is the behaviour of a liquid between two parallel vertical walls with mutual distance $2d > 0$. That is, we have the domain $\Omega = \{-d \leq x_1 \leq d, -\infty < x_2 < \infty\}$ and so $\Sigma = \{x_1 = \pm d, -\infty < x_2 < \infty\}$, see Figure 2.4. Here the calculations lead to a similar

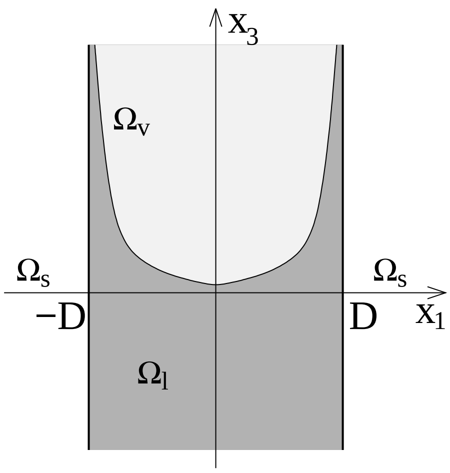

Figure 2.4: Ascent between two parallel plates.

result as above, namely to the following theorem.

Theorem 10
Let v be the solution of (2.2) over the domain $\Omega = \{-d \le x_1 \le d, -\infty < x_2 < \infty\}$. Then there is a constant $A \ge 0$ with

$$\left|v(x_1) - C\cdot(d+x_1)^{3-p} - C\cdot(d-x_1)^{3-p}\right| \le A \qquad \text{in } \Omega,$$

where $C = \dfrac{-2\mathcal{H}}{\pi\kappa\sigma(p-2)(p-3)} > 0.$

That is, for $p = 6$ we have in particular $C = -\dfrac{\mathcal{H}}{6\pi\kappa\sigma} > 0$ and so

$$v \sim -\frac{\mathcal{H}}{6\pi\kappa\sigma}(d \pm x_1)^{-3}, \qquad \text{as } x_1 \to \mp d.$$

2.6 Behaviour at a corner

In this section we will examine the situation that Ω is a vertex, that is, a domain bounded by two straight lines. Without loss of generality, we can choose the domain, such that the cusp lies in the origin, symmetric to the x_1-axis. So we have

$$\Omega = \left\{x \in \mathbb{R}^2 : |x_2| \le x_1 \tan\alpha\right\},$$

whereby $0 < \alpha < \pi/2$. To study the behaviour near the cusp we will examine in addition the solution v of the problem (2.2) over the domain

$$\Omega_\rho = \left\{x \in \mathbb{R}^2 : |x_2| \le x_1 \tan\alpha\right\} \cap B_\rho(0),$$

whereby $B_\rho(0) = \left\{x_1^2 + x_2^2 < \rho^2\right\}$ and $\rho \in (0,1)$, see Figure 2.5. Since it is impossible to calculate the disjoining pressure potential explicitly, we will restrict our examinations to the common case $p = 6$.

After these preparations, we formulate the last theorem.

Theorem 11
Let v be a solution of (2.2) for $p = 6$ over the domain Ω, $0 < \alpha < \pi/2$. Then there are constants $0 < \rho < 1$, $B \in \mathbb{R}$ and $A \ge 0$, independent of the special solution v considered, so that we have

$$\left|v(r,\theta) - \frac{f(\theta)}{r^3} - \frac{h(\theta)}{r}\right| \le A \qquad \text{in } \Omega,$$

and

$$\left|v(r,\theta) - \frac{f(\theta)}{r^3} - \frac{h(\theta)}{r} - B\right| \le A\cdot r \qquad \text{in } \Omega_\rho.$$

Thereby f and h are given by

$$f(\theta) = C\cdot\left(3\cot\frac{\alpha+\theta}{2} + 3\cot\frac{\alpha-\theta}{2} + \cot^3\frac{\alpha+\theta}{2} + \cot^3\frac{\alpha-\theta}{2}\right), \qquad C = \frac{-\mathcal{H}}{16\pi\sigma\kappa} > 0,$$

$$h(\theta) = \frac{3}{\kappa} f(\theta)\frac{3f(\theta)\cdot f''(\theta) - 9f(\theta)^2 - 4f'(\theta)^2}{[9f(\theta)^2 + f'(\theta)^2]^{3/2}}.$$

If $\alpha \in (\pi/2, \pi)$, that is, if the solid domain is a wedge, we get the same result as mentioned in Theorem 11.

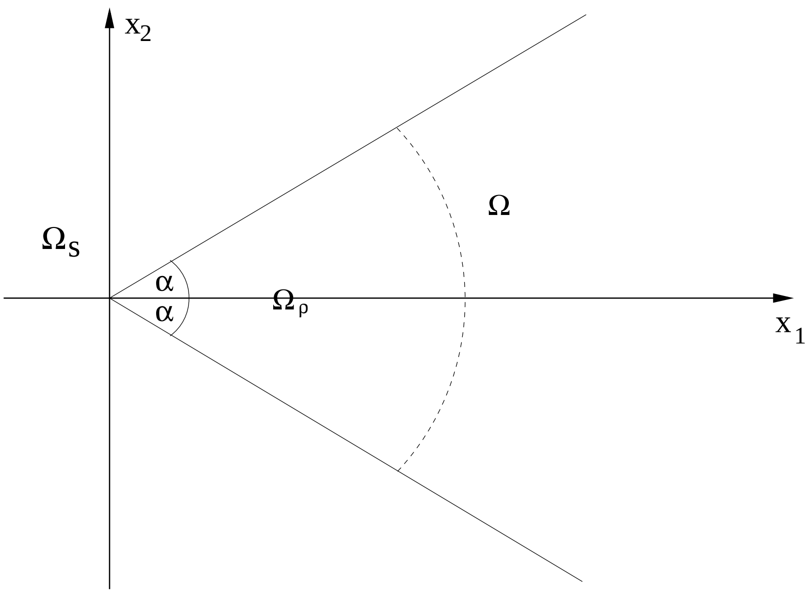

Figure 2.5: The domain Ω_ρ.

Chapter 3

Proofs

Here the omitted proofs of the theorems, stated in the pevious chaper, are given. After proving the Comparison Principles, the modified one will be used when proving the following theorems. The most difficult part to prove the Theorems 8 to 10, is to show the boundedness of div $\mathbf{T}\,P$ in the underlying domains. Then with the help of Theorem 6 it is easy to verify the desired assertions. The proof of Theorem 11 is more complicated, so we have divided it into some subsections, to permit a better comprehension.

3.1 Proof of Theorem 3

The classical Comparison Principle has been proven a lot of times before for a variety of situations ($\kappa > 0$, $\kappa = 0$, bounded or unbounded domains, ...). We give the proofe, it because we need it for the special case of unbounded domains $\Omega \subset \mathbb{R}^2$. The other reason for showing the proof is therewith that it is easier to outline the proof of the modified Comparison Principle.

Theorem 3 (Normal Comparison Principle)
Let $\kappa > 0$ and suppose $\Sigma = \partial\Omega$ admits a decomposition $\Sigma = \Sigma_\alpha \cup \Sigma_\beta \cup \Sigma_0$, where $\Sigma_\beta \in \mathcal{C}^1$ and Σ_0 has one-dimensional Hausdorff measure zero. Let $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\Sigma_\beta \cup \Omega)$ with the properties

(i)	$\operatorname{div} \boldsymbol{T}u - \kappa u$	$\geq \operatorname{div} \boldsymbol{T}v - \kappa v$	*in* Ω,
(ii)	u	$\leq v$	*as* $x \to \Sigma_\alpha$,
(iii)	$\nu \cdot \boldsymbol{T}u$	$\leq \nu \cdot \boldsymbol{T}v$	*as* $x \to \Sigma_\beta$.

Then $v \geq u$ in Ω.

Proof:
We follow the proofs given in [Fin86, p.111] and [FH89].

For any $R > 0$, we set $B_R = \{x \in \mathbb{R}^2 : |x| < R\}$, $\Omega_R = \Omega \cap B_R$, $\Gamma_R = \partial B_R \cap \Omega$, see Figure 3.1.

Suppose the proposition is not correct. That is, suppose u and v satisfy (i), (ii), (iii) and for some $x_0 \in \Omega$ it holds $u(x_0) - v(x_0) > 0$. For some positive M we define the domains

$$\tilde{\Omega}_1 = \{x \in \Omega : u - v < 0\},$$
$$\tilde{\Omega}_2 = \{x \in \Omega : 0 < u - v < M\},$$
$$\tilde{\Omega}_3 = \{x \in \Omega : u - v > M\},$$

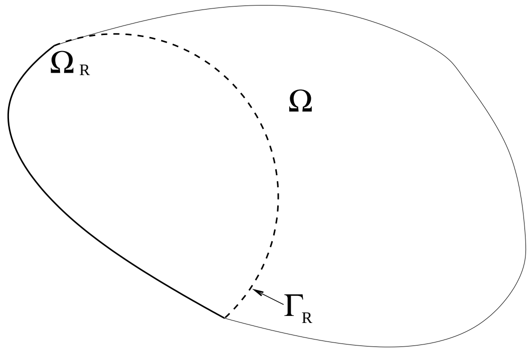

Figure 3.1: Definition of Ω_R and Γ_R.

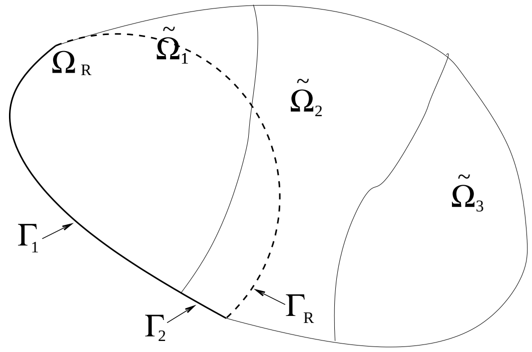

Figure 3.2: Partition of Ω.

and for $i = 1, 2, 3$ we set

$$\Omega_i = \tilde{\Omega}_i \cap B_R \text{ and}$$
$$\Gamma_i = \partial\Omega \cap \partial\Omega_i \cap B_R,$$

see Figure 3.2. Due to our assumptions we can choose M and R so that $|\Omega_2|$ has positive measure. Let

$$w(x) = \begin{cases} 0 & \text{, in } \Omega_1 \\ u - v & \text{, in } \Omega_2 \\ M & \text{, in } \Omega_3. \end{cases}$$

The basic idea of the proof is to show that the integral

$$\int_{\Omega_R} w \cdot ([\operatorname{div} \mathbf{T}\, u - \kappa u] - [\operatorname{div} \mathbf{T}\, v - \kappa v])\mathrm{d}x$$

vanishes, which leads to a contradiction and proves the theorem. Using integration by parts we obtain the following estimates:

$$\begin{aligned} 0 &\le \int_{\Omega_R} w \cdot ([\operatorname{div} \mathbf{T}\, u - \kappa u] - [\operatorname{div} \mathbf{T}\, v - \kappa v])\mathrm{d}x \\ &= \int_{\Omega_R} w \cdot (\operatorname{div} \mathbf{T}\, u - \operatorname{div} \mathbf{T}\, v)\mathrm{d}x - \kappa \int_{\Omega_R} w \cdot (u - v)\mathrm{d}x \\ &= -\int_{\Omega_R} \nabla w \cdot (\mathbf{T}\, u - \mathbf{T}\, v)\,\mathrm{d}x - \kappa \int_{\Omega_R} w \cdot (u - v)\mathrm{d}x + \int_{\partial\Omega_R} w \cdot (\mathbf{T}\, u - \mathbf{T}\, v) \cdot \nu \mathrm{d}s. \\ &= -\int_{\Omega_2} \nabla(u - v) \cdot (\mathbf{T}\, u - \mathbf{T}\, v)\,\mathrm{d}x - \kappa \int_{\Omega_R} w \cdot (u - v)\mathrm{d}x + \int_{\partial\Omega_R} w \cdot (\mathbf{T}\, u - \mathbf{T}\, v) \cdot \nu \mathrm{d}s, \end{aligned} \tag{3.1}$$

or, assigning in symbols to the integrals on the right in order of appearance,

$$0 \le -Q - W + R. \tag{3.2}$$

With regard to Q, write $\nabla u = p$, $\nabla v = q$ and $\mathbf{T}\, u = \mathbf{A}(p)$. Consider the function

$$\mathcal{F}(t) = (p - q) \cdot [\mathbf{A}(q + t(p - q)) - \mathbf{A}(q)], \qquad 0 \le t \le 1.$$

Then we have $\mathcal{F}(0) = 0$ and $\mathcal{F}(1) = \nabla(u - v) \cdot (\mathbf{T}\, u - \mathbf{T}\, v)$. Since

$$\mathcal{F}'(t) = (p - q) \cdot \frac{\partial}{\partial t} \frac{q + t(p - q)}{\sqrt{1 + |q + t(p - q)|^2}} = \frac{|p - q|^2}{(1 + |q + t(p - q)|^2)^{3/2}} \geq 0$$

we can conclude $\nabla(u - v)(\mathbf{T}\, u - \mathbf{T}\, v) = \mathcal{F}(1) \geq \mathcal{F}(0) = 0$ in Ω_2 and so $Q \geq 0$.
In the next we examine the Integral R:

$$\begin{aligned} R &= \int_{\Gamma_1} w(\mathbf{T}\, u - \mathbf{T}\, v) \cdot \nu \mathrm{d}s + \int_{\Gamma_2} w(\mathbf{T}\, u - \mathbf{T}\, v) \cdot \nu \mathrm{d}s + \int_{\Gamma_3} w(\mathbf{T}\, u - \mathbf{T}\, v) \cdot \nu \mathrm{d}s \\ &\quad + \int_{\Gamma_R} w(\mathbf{T}\, u - \mathbf{T}\, v) \cdot \nu \mathrm{d}s \\ &= \int_{\Gamma_2} (u - v)(\mathbf{T}\, u - \mathbf{T}\, v) \cdot \nu \mathrm{d}s + M \int_{\Gamma_3} (\mathbf{T}\, u - \mathbf{T}\, v) \cdot \nu \mathrm{d}s + \int_{\Gamma_R} w(\mathbf{T}\, u - \mathbf{T}\, v) \cdot \nu \mathrm{d}s \\ &\equiv I_2 + I_3 + I_R. \end{aligned}$$

By assumption we have $\Gamma_2 \subset \Gamma_\beta$ and $\Gamma_3 \subset \Gamma_\beta$, hence $I_2 \leq 0$ and $I_3 \leq 0$.
So (3.2) reduces to $0 \leq I_R - W$ that is

$$\kappa \int_{\Omega_R} w(u - v) \mathrm{d}x \leq \int_{\Gamma_R} w(\mathbf{T}\, u - \mathbf{T}\, v) \nu \mathrm{d}s \tag{3.3}$$

which will lead to the desired contradiction.
Since $w^2 \leq w(u - v)$ in Ω_R and $\mathbf{T}\, u$ is bounded by 1 for every u, we conclude from (3.3)

$$\kappa \int_{\Omega_R} w^2 \mathrm{d}x \leq \kappa \int_{\Omega_R} w(u - v) \mathrm{d}x \leq \int_{\Gamma_R} w(\mathbf{T}\, u - \mathbf{T}\, v) \nu \mathrm{d}s \leq 2 \int_{\Gamma_R} w \nu \mathrm{d}s = 2 \int_{\Gamma_R} w \mathrm{d}S.$$

So, we get with Hölder's inequality

$$Q(R) \equiv \int_{\Omega_R} w^2 \mathrm{d}x \leq \frac{2}{\kappa} \left(\int_{\Gamma_R} w^2 \mathrm{d}S \right)^{\frac{1}{2}} \left(\int_{\Gamma_R} \mathrm{d}S \right)^{\frac{1}{2}} \leq \sqrt{C \cdot R} \left(\int_{\Gamma_R} w^2 \mathrm{d}S \right)^{\frac{1}{2}}$$

for some positive C, which is equivalent to

$$\frac{1}{R} Q^2(R) \leq C \int_{\Gamma_R} w^2 \mathrm{d}S. \tag{3.4}$$

Integrating (3.4) yields

$$\begin{aligned} J(R) &\equiv \int_{R_1}^{R} \frac{1}{\rho} Q^2(\rho) \mathrm{d}\rho \leq C \int_{R_1}^{R} \int_{\Gamma_\rho} w^2 \mathrm{d}S \mathrm{d}\rho = C \int_{\Omega_R \setminus \Omega_{R_1}} w^2 \mathrm{d}x \\ &= C \int_{\Omega_R} w^2 \mathrm{d}x - C \int_{\Omega_{R_1}} w^2 \mathrm{d}x = C \left[Q(R) - Q(R_1) \right]. \end{aligned}$$

If R_1 is sufficiently small, then $J(R) > 0$ and we conclude

$$\frac{J'(R)}{J^2(R)} = \frac{Q^2(R)}{R J(R)} \geq \frac{Q^2(R)}{C^2 R [Q(R) - Q(R_1)]^2} \geq \frac{\tilde{C}}{R} \tag{3.5}$$

for all sufficiently large R. Again, by integrating (3.5) we have

$$\frac{1}{J(R_1)} - \frac{1}{J(R)} = \int_{R_1}^{R} \frac{J'(\rho)}{J^2(\rho)} \mathrm{d}\rho \geq \int_{R_1}^{R} \frac{\tilde{C}}{R} \mathrm{d}\rho = \tilde{C} \ln(R/R_1).$$

For sufficiently large R, this leads to a contradiction and establishes $u \leq v$ in Ω. □

3.2 Proof of Theorem 5

This theorem is an elementary extension of the previous one. Its proof is also based on the preceding one.

Theorem 5 (Comparison principle)
Let $\kappa > 0$ *and suppose* $\Sigma = \partial\Omega$ *admits a decomposition* $\Sigma = \Sigma_\alpha \cup \Sigma_\beta \cup \Sigma_0$*, where* $\Sigma_\beta \in \mathcal{C}^1$ *and* Σ_0 *has one-dimensional Hausdorff measure zero. Let* $u, v \in \mathcal{C}^2(\Omega)$ *with the properties*

(i)	$\operatorname{div} \boldsymbol{T}u - \kappa u - P(x,u)$	$\geq \operatorname{div} \boldsymbol{T}v - \kappa v - P(x,v)$	in Ω,
(ii)	u	$\leq v$	as $x \to \Sigma_\alpha$,
(iii)	$\nu \cdot \boldsymbol{T}u$	$\leq \nu \cdot \boldsymbol{T}v$	as $x \to \Sigma_\beta$.

Then we have $u \leq v$ *in* Ω.

Proof:
The proof is based on the proof of Theorem 3. With the same notations as in the mentioned proof we now are led to the inequality

$$0 \leq -Q - W + R - G,$$

see (3.1) and (3.2) respectively for the definition of Q, W and R. The new arising integral G is

$$G = \int_{\Omega_R} w \cdot (P(x,u) - P(x,v))\mathrm{d}x.$$

To show the desired estimate, it is sufficient to show the non-negativity of the integrand. To prove this, we prove the monotonicity of $P(x,u)$ in u that is, we show $\frac{\partial P}{\partial u} \geq 0$:

$$\begin{aligned}
\frac{\partial P}{\partial u} &= c \int_{\Omega_s} \frac{\partial}{\partial u} \left[(x_1-y_1)^2 + (x_2-y_2)^2 + (u-y_3)^2\right]^{-p/2} \mathrm{d}y \\
&= -pc \int_{\Omega_s} (u-y_3) \left[(x_1-y_1)^2 + (x_2-y_2)^2 + (u-y_3)^2\right]^{-p/2-1} \mathrm{d}y \\
&= -pc \int_{\Omega} \left(\int_{-\infty}^{0} (u-y_3) \left[(x_1-y_1)^2 + (x_2-y_2)^2 + (u-y_3)^2\right]^{-p/2-1} \mathrm{d}y_3 \right) \mathrm{d}y_1 \mathrm{d}y_2 \\
&\quad - pc \int_{\mathbb{R}^2 \setminus \Omega} \left(\int_{-\infty}^{\infty} (u-y_3) \left[(x_1-y_1)^2 + (x_2-y_2)^2 + (u-y_3)^2\right]^{-p/2-1} \mathrm{d}y_3 \right) \mathrm{d}y_1 \mathrm{d}y_2 \\
&= -pc \int_{\Omega} \left(\int_{-\infty}^{0} (u-y_3) \left[(x_1-y_1)^2 + (x_2-y_2)^2 + (u-y_3)^2\right]^{-p/2-1} \mathrm{d}y_3 \right) \mathrm{d}y_1 \mathrm{d}y_2 \\
&\geq 0,
\end{aligned}$$

since c is negative and u is positive. So we have $P(x,u) - P(x,v) \geq 0$ for $u \geq v$. Consequently we get

$$G = \int_{\Omega_2} (u-v)[P(x,u) - P(x,v)]\mathrm{d}x + M \int_{\Omega_3} [P(x,u) - P(x,v))]\mathrm{d}x \geq 0.$$

Thus the theorem is proven. □

3.3 Proof of Lemma 7

Lemma 7
Let be $\Omega \subset \mathbb{R}^2$ and let v define a capillary surface over the domain Ω, so that (2.2) holds with $\kappa > 0$.
Then v is bounded on every compact subset of Ω.

Proof:
We follow the proof given in [Fin86, Theorem 5.2].

We use polar coordinates (ρ, θ), to desrcibe the points of the compact subset. Let B_δ be a disc of radius $\delta > 0$ with $B_\delta \Subset \Omega$[1] with centre $x = (\rho \cos \theta_0,\, \rho \sin \theta_0)$ and radius δ, see Figure 3.3. Let T_δ be the trapezium, bounded by $t_1, \ldots, t_4$. Here $t_1, \ldots, t_4$ are the tangents on B_δ, crossing the origin and the normals to their bisecting line respectively, see Figure 3.4.

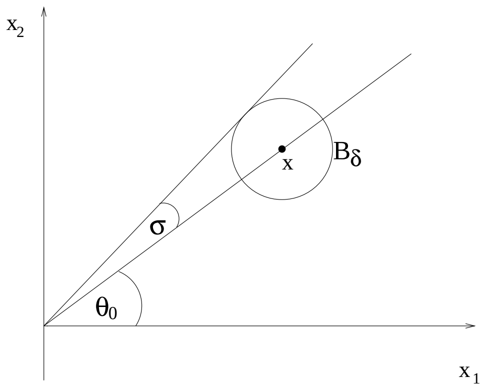

Figure 3.3: A disc on the arc Γ_ρ.

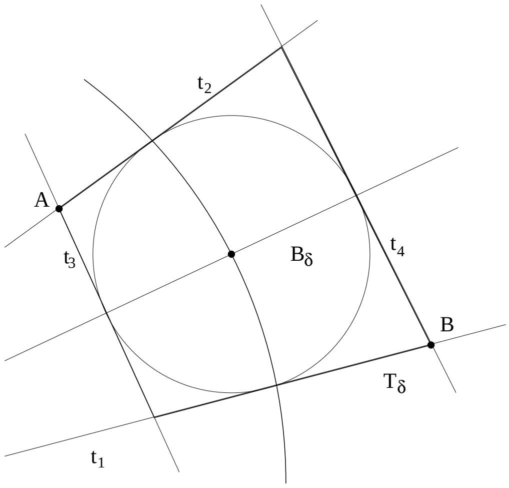

Figure 3.4: The trapezium T_δ.

Then we use the definition of $P(x)$ to estimate for ρ fixed

$$\sup_{x \in B_\delta} |P(x, u(x))| \leq \sup_{x \in T_\delta} |P(x, u(x))| = |P(A, u(A))| < \infty,$$

where A is the intersection between t_2 and t_3.
Let u denote a lower hemisphere over B_δ whose lowest point has the height

$$u_0 = \frac{2}{\kappa\delta} - \frac{P(A, u(A))}{\kappa}.$$

Since $P(x, u)$ is always negative, we have

$$\operatorname{div} \mathbf{T}\, u = 2H = \frac{2}{\delta} = \kappa u_0 + P(A, u(A)) \leq \kappa u + P(x, u(x)) \qquad \text{in } B_\delta.$$

[1]Let A and B be open sets with $A \subseteq B$, then we define $A \Subset B :\Leftrightarrow \bar{A} \subseteq B$.

That is, we obtain

$$\begin{aligned} \operatorname{div} \mathbf{T}\, u - \kappa u + P \leq 0 \qquad & \text{in } B_\delta, \\ \nu \cdot \mathbf{T}\, u = 1 \qquad & \text{on } \partial B_\delta. \end{aligned}$$

Since we have for a solution of equation (2.2)

$$\begin{aligned} \operatorname{div} \mathbf{T}\, v - \kappa v - P = 0 \qquad & \text{in } B_\delta, \\ \nu \cdot \mathbf{T}\, v \leq 1 \qquad & \text{on } \partial B_\delta, \end{aligned}$$

we apply Theorem 5, with $\Sigma_\beta = \partial B_\delta$ and $\Sigma_\alpha = \Sigma_0 = \emptyset$ to get

$$v < u \leq \frac{2}{\kappa\delta} + \delta - \frac{P(A, u(A))}{\kappa} < \infty.$$

Since we can cover every compact subset by balls, the lemma is proven. □

3.4 Proof of Theorem 8

Theorem 8
Let v be the solution of problem (2.2) over the domain $B_1(0)$. Then there is a constant $A \geq 0$, with

$$\left| v(q) - C \cdot \mathrm{F}(a, b; 1; q^2) \right| \leq A, \qquad \text{in } B_1(0),$$

where $a = \frac{p-3}{2}$, $b = \frac{p-1}{2}$, $C = -\pi^{-\frac{1}{2}} \mathcal{H} \kappa^{-1} \sigma^{-1} R^{3-p} \Gamma(a)/\Gamma(p/2) > 0$ and $\mathrm{F}(a, b; 1; q^2)$ is the hypergeometric Function.

Proof:
The idea of the proof is to show that $P(x, u)$ satisfies the conditions of Theorem 6. To verify these conditions, it is necessary to compute $P(x, u)$.

In the present symmetric case, it is helpful to introduce polar and cylindrical coordinates for (x_1, x_2) and (y_1, y_2, y_3) respectively. We set $x_1 = Rq\cos\theta$, $x_2 = Rq\sin\theta$ and $y_1 = Rs\cos\varphi$, $y_2 = Rs\sin\varphi$, $y_3 = Rh$. Henceforward we denote $r = \sqrt{x_1^2 + x_2^2}$ and $q = r/R$. At this juncture, r and q are the distance to the origin in $B_R(0)$ and $B_1(0)$ respectively. Then we define $U(q) = u(x_1, x_2)$, $K = K(q, \theta, U(q)) = P(x, u(x_1, x_2))$ and $c_0 = \mathcal{H}\sigma^{-1}\pi^{-2}R^{3-p}$ and thus we can compute the disjoining pressure potential

$$\begin{aligned} K &= c_0 \int_1^\infty \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} s \left[(q\cos\theta - s\cos\varphi)^2 + (q\sin\theta - s\sin\varphi)^2 + (U/R - h)^2 \right]^{-p/2} \mathrm{d}h \mathrm{d}\varphi \mathrm{d}s \\ &= c_0 \int_1^\infty \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} s \left[(q\cos\theta - s\cos\varphi)^2 + (q\sin\theta - s\sin\varphi)^2 + h^2 \right]^{-p/2} \mathrm{d}h \mathrm{d}\varphi \mathrm{d}s \\ &= c_0 \int_1^\infty \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} s \left[s^2 - 2qs(\cos\varphi\cos\theta + \sin\varphi\sin\theta) + q^2 + h^2 \right]^{-p/2} \mathrm{d}h \mathrm{d}\varphi \mathrm{d}s \\ &= c_0 \int_1^\infty \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} s \left[s^2 - 2qs\cos(\varphi - \theta) + q^2 + h^2 \right]^{-p/2} \mathrm{d}h \mathrm{d}\varphi \mathrm{d}s \\ &= c_0 \int_1^\infty \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} s \left[s^2 - 2qs\cos\varphi + q^2 + h^2 \right]^{-p/2} \mathrm{d}h \mathrm{d}\varphi \mathrm{d}s \\ &= 2c_0 \int_1^\infty \int_0^{\pi} \int_{-\infty}^{\infty} s \left[s^2 - 2qs\cos\varphi + q^2 + h^2 \right]^{-p/2} \mathrm{d}h \mathrm{d}\varphi \mathrm{d}s. \end{aligned}$$

The values of θ and U can be omitted in the integral due to the integration limits. It turns out that the disjoining pressure potential is a function depending only on one parameter, q. In detail we have

$$P(x, u(x)) = c_0 g(q),$$

whereby $c_0 = \mathcal{H}\sigma^{-1}\pi^{-2}R^{3-p}$, $q = r/R$ and $r = \sqrt{x_1^2 + x_2^2}$ is the distance to the origin and

$$g(q) = 2\int_{\Omega_1} s\left[s^2 - 2qs\cos\varphi + q^2 + h^2\right]^{-p/2} \mathrm{d}h\mathrm{d}\varphi\mathrm{d}s.$$

The domain of integration is $\Omega_1 = \{(s, \varphi, h) \in \mathbb{R}^3 : \ 1 < s < \infty, \ 0 < \varphi < \pi, \ -\infty < h < \infty\}$. Furthermore we follow the achievements of PHILIP in [Phi77, IV]. The substitution $h = [s^2 - 2sq\cos\varphi + q^2]^{1/2} \cdot \tan t$ for $-\pi/2 < t < \pi/2$ leads to the Beta Function $\mathrm{B}(.,.)$ that is, we obtain with the help of [GR82, Formula 3.621]

$$\begin{aligned} g(q) &= 4\int_1^\infty \int_0^\pi s\left[s^2 - 2sq\cos\varphi + q^2\right]^{\frac{1-p}{2}} \mathrm{d}\varphi\mathrm{d}s \int_0^{\pi/2} (\cos t)^{p-2}\mathrm{d}t \\ &= 2^{p-1}\mathrm{B}\left(\frac{p-1}{2}, \frac{p-1}{2}\right) \int_1^\infty \int_0^\pi s\left[s^2 - 2sq\cos\varphi + q^2\right]^{\frac{1-p}{2}} \mathrm{d}\varphi\mathrm{d}s \\ &= 2^{p-1}\frac{\Gamma^2(\frac{p-1}{2})}{\Gamma(p-1)} \int_1^\infty \int_0^\pi s\left[s^2 - 2sq\cos\varphi + q^2\right]^{\frac{1-p}{2}} \mathrm{d}\varphi\mathrm{d}s. \end{aligned}$$

By integrating over φ, the hypergeometric function $\mathrm{F}(a, b; c; z)$ appears. For handling hypergeometric functions see for example [AS64, p. 555 ff] or [Kle33]. In detail we obtain

$$\begin{aligned} g(q) &= 2^{p-1}\frac{\Gamma^2(\frac{p-1}{2})}{\Gamma(p-1)} \int_1^\infty s^{2-p} \int_0^\pi \left[1 - 2\frac{q}{s}\cos\varphi + \frac{q^2}{s^2}\right]^{\frac{1-p}{2}} \mathrm{d}\varphi\mathrm{d}s \\ &= 2^{p-1}\frac{\Gamma^2(\frac{p-1}{2})}{\Gamma(p-1)}\mathrm{B}\left(\frac{1}{2}, \frac{1}{2}\right) \int_1^\infty s^{2-p}\mathrm{F}\left(\frac{p-1}{2}, \frac{p-1}{2}; 1; \left(\frac{q}{s}\right)^2\right) \mathrm{d}s \\ &= 2\pi^{\frac{3}{2}}\frac{\Gamma(\frac{p-1}{2})}{\Gamma\left(\frac{p}{2}\right)} \int_1^\infty s^{2-p}\mathrm{F}\left(\frac{p-1}{2}, \frac{p-1}{2}; 1; \left(\frac{q}{s}\right)^2\right) \mathrm{d}s, \end{aligned}$$

where we have used the relation

$$\int_0^\pi \frac{\sin^{2\lambda-1} x\mathrm{d}x}{(1 + 2a\cos x + a^2)^\nu} = \mathrm{B}\left(\lambda, \frac{1}{2}\right)\mathrm{F}\left(\nu, \nu - \lambda + \frac{1}{2}; \lambda + \frac{1}{2}; a^2\right),$$

see [GR82, 3.665.2], with $\lambda = 1/2$ and $\nu = (p-1)/2$, $\mathrm{B}\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$ and the formula

$$\frac{\Gamma(z)}{\Gamma(2z)} = \frac{\sqrt{\pi} \cdot 2^{1-2z}}{\Gamma\left(z + \frac{1}{2}\right)}.$$

Now, using the transformation $t = q^2/s^2$, for $q^2 > t > 0$, we get

$$
\begin{aligned}
g(q) &= \pi^{\frac{3}{2}} \frac{\Gamma(\frac{p-1}{2})}{\Gamma\left(\frac{p}{2}\right)} \int_0^{q^2} s^{5-p} q^{-2} \mathrm{F}\left(\frac{p-1}{2}, \frac{p-1}{2}; 1; t\right) \mathrm{d}t \\
&= \pi^{\frac{3}{2}} \frac{\Gamma\left(\frac{p-1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} q^{3-p} \int_0^{q^2} t^{\frac{p-5}{2}} \mathrm{F}\left(\frac{p-1}{2}, \frac{p-1}{2}; 1; t\right) \mathrm{d}t \\
&= 2\pi^{\frac{3}{2}} \frac{\Gamma\left(\frac{p-1}{2}\right)}{(p-3)\Gamma\left(\frac{p}{2}\right)} \mathrm{F}\left(\frac{p-3}{2}, \frac{p-1}{2}; 1; q^2\right). \\
&= \pi^{\frac{3}{2}} \frac{\Gamma\left(\frac{p-3}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \mathrm{F}\left(\frac{p-3}{2}, \frac{p-1}{2}; 1; q^2\right).
\end{aligned}
\tag{3.6}
$$

In formula (3.6) we used

$$
\frac{d}{dz}\left[z^{a-1}\mathrm{F}(a-1, b; c; z)\right] = (a-1)z^{a-2}\mathrm{F}(a, b; c, z),
$$

see [AS64, 15.2.3].
So we found for the case of the infinite tube that the disjoining pressure potential becomes

$$
P(x, u(x)) = C \cdot \mathrm{F}\left(a, b; 1; q^2\right), \quad \text{with } a = \frac{p-3}{2} \text{ and } b = \frac{p-1}{2},
$$

$q = \sqrt{x_1^2 + x_2^2}/R$ and $C = \mathcal{H}\sigma^{-1}\pi^{-\frac{1}{2}}R^{3-p}\Gamma(\frac{p-3}{2})/\Gamma(\frac{p}{2})$.
For the sake of simplicity, we set $\psi = \psi(q) = \mathrm{F}(a, b; 1; q^2)$. To apply Theorem 6 it is sufficient to verify that $\operatorname{div} \mathbf{T}\,\psi$ is bounded. When using polar coordinates it is necessary to show that

$$
|\operatorname{div} \mathbf{T}\,\psi| = \left|\frac{1}{q}\left(\frac{q\psi'}{\sqrt{1+\psi'^2}}\right)'\right| = \left|\frac{\psi'}{q\sqrt{1+\psi'^2}} + \frac{\psi''}{(1+\psi'^2)^{3/2}}\right| \leq K < \infty,
$$

for all $q \in [0, 1]$. At first we present the derivatives of $\psi(q)$ by using formula [AS64, 15.2.1]:

$$
\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{F}(a, b; c; z) = \frac{ab}{c}\mathrm{F}(a+1, b+1; c+1; z).
$$

To state the derivatives in an appropriate form we use formula [AS64, 15.3.6]

$$
\begin{aligned}
\mathrm{F}(a, b; c; z) = &\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\mathrm{F}(a, b; a+b-c+1; 1-z) \\
&+ (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\mathrm{F}(c-a, c-b; c-a-b+1; 1-z).
\end{aligned}
$$

So we separate the highest poles in the according formulae. In these cases the constants $A_1, \ldots, A_6$ occuring the in equations (3.7) and (3.8) are explicitly known positive constants, depending only on p. In detail we have

$$
\begin{aligned}
\psi'(q) &= 2abq\mathrm{F}(a+1, b+1; 2; q^2) \\
&= A_1 q\mathrm{F}(a+1, b+1; a+b; 1-q^2) \\
&\quad + A_2 q\left(1-q^2\right)^{-a-b}\mathrm{F}\left(1-a, 1-b; 1-a-b; 1-q^2\right)
\end{aligned}
\tag{3.7}
$$

and

$$\begin{aligned}\psi''(q) &= 2ab\mathrm{F}(a+1,b+1;2;q^2) + 2ab(a+1)(b+1)q^2\mathrm{F}\left(a+2,b+2;3;q^2\right)\\ &= A_3\mathrm{F}(a+1,b+1;a+b+1;1-q^2) + A_4q^2\mathrm{F}\left(a+2,b+2;a+b+2;1-q^2\right)\\ &\quad + A_5\left(1-q^2\right)^{-a-b}\mathrm{F}\left(1-a,1-b;1-a-b;1-q^2\right)\\ &\quad + A_6q^2\left(1-q^2\right)^{-1-a-b}\mathrm{F}\left(1-a,1-b;-a-b;1-q^2\right).\end{aligned} \tag{3.8}$$

In the case of $a+b\in\mathbb{Z}$ we need a slight modification of equations (3.7) and (3.8) respectively. Thereby we use formulae [AS64, 15.3.10-12] instead of [AS64, 15.3.6]. This modification yields the same result, which can be confirmed by an easy calculation. However, the result is that if q tends to 1, the leading singularities in both cases are of the same order.
In the first parts in formulae (3.7) and (3.8) respectively we can see that both ψ' and ψ'' are bounded for $q\in\left[0,\frac{1}{2}\right]$. In particular we have got

$$\left|\frac{\psi'}{q\sqrt{1+\psi'^2}}\right| \le \frac{|\psi'|}{q} = 2ab|\mathrm{F}\left(a+1,b+1;2;q^2\right)| < K$$

and

$$\begin{aligned}\left|\frac{\psi''}{(1+\psi'^2)^{3/2}}\right| \le |\psi''| &\le 2ab|\mathrm{F}\left(a+1,b+1;2;q^2\right)|\\ &\quad + 2ab(a+1)(b+1)q^2|\mathrm{F}\left(a+2,b+2;3;q^2\right)| < K,\end{aligned}$$

for a constant K with $0\le K<\infty$.
To prove the estimate for $q\in\left(\frac{1}{2},1\right]$, we use the second parts of formulae (3.7) and (3.8) respectively. In this case we get

$$\left|\frac{\psi'}{q\sqrt{1+\psi'^2}}\right| \le 2\frac{\psi'}{\sqrt{1+\psi'^2}} \le 2,$$

and if $\mathrm{F}_1,\dots,\mathrm{F}_6$ denote the according hypergeometric functions in formulae (3.7) and (3.8)

$$\left|\frac{\psi''}{(1+\psi'^2)^{3/2}}\right| \le \left(1-q^2\right)^{2a+2b-1}\frac{(1-q^2)^{1+a+b}|A_3\mathrm{F}_3+A_4q^2\mathrm{F}_4| + (1-q^2)A_5|\mathrm{F}_5| + A_6q^2|\mathrm{F}_6|}{\left[(1-q^2)^{2a+2b} + q^2\left\{A_1\left(1-q^2\right)^{a+b}\mathrm{F}_1 + A_2\mathrm{F}_2\right\}^2\right]^{3/2}}.$$

The right-hand-side of the equation above is bounded. This is easy to see when we remember that $A_3,\dots,A_6$, $\mathrm{F}_3,\dots,\mathrm{F}_6$ are bounded for $q\in(\frac{1}{2},1]$ and since $2a+2b-1$ is positive.
So we can apply Theorem 6 and Theorem 8 is proven. □

Let us look at the case of the closed tube. That is, we have

$$\Omega_s = \mathbb{R}^3\setminus\{y\in\mathbb{R}^3 : y_1^2+y_2^2\le R^2,\ y_3\ge 0\}.$$

Let be in addition c a positive constant with $v(x)\ge c$ for all $x\in\Omega$. Thereby v denotes the solution of (2.2). So we have

$$\int_{\Omega_2}[(x_1-y_1)^2+(x_2-y_2)^2+(u-y_3)^2]^{-p/2}dy \le K_1\int_{-\infty}^{0}\frac{\mathrm{d}y_3}{|u-y_3|^p} \le K_1\int_c^\infty\frac{\mathrm{d}y_3}{y_3^p} \le K_2<\infty,$$

where $\Omega_2=\{y\in\mathbb{R}^3 : y_1^2+y_2^2<R^2,\ y_3<0\}$, K_1 and K_2 are some positive constants, depending only on R. So we can adopt the idea of the proof of Theorem 6 with a constant $A+K_2$ instead of A. This establishes the annex of Theorem 8.

3.5 Proof of Theorem 9 and Theorem 10

Theorem 9

Let v be the solution of (2.2) *over the domain* $\Omega = \{x_1 > 0, -\infty < x_2 < \infty\}$. *Then there is a constant* $A \geq 0$ *with*

$$\left| v(x_1) - C \cdot x_1^{3-p} \right| \leq A \qquad \text{in } \Omega,$$

where $C = \dfrac{-2\mathcal{H}}{\pi\kappa\sigma(p-2)(p-3)} > 0$.

Proof:

In order to apply Theorem 6 we have to compute the disjoining pressure potential and check the boundedness of $\operatorname{div} \mathbf{T}\, P$. Because of the special symmetry it is sufficient to look for a solution v of (2.2), which depends only on the distance to the $x_2 - x_3-$plane. As seen in the case of the bottomless cylinder, the disjoining pressure potential is independent of the special solution. Here the solid domain is given by

$$\Omega_s = \left\{ y \in \mathbb{R}^3 : \, y_1 < 0, \, -\infty < y_2 < \infty, \, -\infty < y_3 < \infty \right\}.$$

First of all we compute the disjoining pressure potential. To do this we examine the following integral

$$\begin{aligned} g(x_1) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{0} \left[(x_1 - y_1)^2 + (x_2 - y_2)^2 + (u - y_3)^2 \right]^{-\frac{p}{2}} \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{0} \left[(x_1 - y_1)^2 + y_2^2 + y_3^2 \right]^{-\frac{p}{2}} \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3. \end{aligned}$$

To compute the integral we use cylindrical coordinates,

$$\begin{aligned} y_1 &= x_1 - h, \quad && \infty > h > x_1, \\ y_2 &= r\cos\varphi, \quad && 0 < r < \infty, \\ y_3 &= r\sin\varphi, \quad && 0 < \varphi < \pi, \end{aligned}$$

where the Jacobian is $-r$. If we use the transformation $r = h\tan\alpha$, $0 < \alpha < \frac{\pi}{2}$, we get

$$\begin{aligned} g(x_1) &= \int_{x_1}^{\infty} \int_0^{2\pi} \int_0^{\infty} r \left[h^2 + r^2 \right]^{-\frac{p}{2}} \mathrm{d}r \mathrm{d}\varphi \mathrm{d}h \\ &= 2\pi \int_{x_1}^{\infty} \int_0^{\frac{\pi}{2}} h^2 \left[h^2 + h^2 \frac{\sin^2\alpha}{\cos^2\alpha} \right]^{-\frac{p}{2}} \frac{\sin\alpha}{\cos^3\alpha} \mathrm{d}\alpha \mathrm{d}h \\ &= 2\pi \int_{x_1}^{\infty} h^{2-p} \mathrm{d}h \int_0^{\frac{\pi}{2}} (\cos\alpha)^{p-3} \sin\alpha \mathrm{d}\alpha \\ &= \frac{2\pi}{(p-2)(p-3)} x_1^{3-p}. \end{aligned}$$

To apply Theorem 6 we have to estimate $\operatorname{div} \mathbf{T}\, g$, whereas $\mathbf{T}\, g = \frac{g'}{\sqrt{1+g'^2}}$. With $c_0 = 2\pi(p-2)^{-1}(p-3)^{-1}$, $c_1 = 2\pi(2-p)^{-1}$ and $c_2 = 2\pi$ we get

$$\begin{aligned} g(x_1) &= c_0 x_1^{3-p}, \\ g'(x_1) &= c_1 x_1^{2-p}, \\ g''(x_1) &= c_2 x_1^{1-p}. \end{aligned}$$

So we transform $\operatorname{div} \mathbf{T}\, g$:

$$
\begin{aligned}
\operatorname{div} \mathbf{T}\, g &= \left(\frac{g'}{\sqrt{1+g'^2}} \right)' = g'' \cdot \left(1+g'^2\right)^{-\frac{3}{2}} \\
&= c_2 x_1^{1-p} \cdot \left\{ 1 + c_1^2 x_1^{4-2p} \right\}^{-\frac{3}{2}} \\
&= c_2 x_1^{2p-5} \cdot \left\{ x_1^{2p-4} + c_1^2 \right\}^{-\frac{3}{2}} .
\end{aligned}
\tag{3.9}
$$

Since $\operatorname{div} \mathbf{T}\, g$ is continuous for $x_1 \in (0, \infty)$ it is sufficient to show that $\operatorname{div} \mathbf{T}\, g$ is bounded, as x_1 tends to 0 and ∞ respectively.
From the second line in equation (3.9) we conclude that $\operatorname{div} \mathbf{T}\, g$ tends to 0 as x_1 tends to infinity. We get the same result if x_1 tends to 0 from the last identity in formula (3.9). That means the expression $\operatorname{div} \mathbf{T}\, g$ is bounded for all $x_1 \in (0, \infty)$ and so $\operatorname{div} \mathbf{T}\, P$ is bounded, too.
So the theorem is proven. □

Theorem 10
Let v be the solution of (2.2) over the domain $\Omega = \{-d \le x_1 \le d, -\infty < x_2 < \infty\}$. Then there is a constant $A \ge 0$ with

$$
\left| v(x_1) - C \cdot (d + x_1)^{3-p} - C \cdot (d - x_1)^{3-p} \right| \le A \qquad \textit{in } \Omega,
$$

where $C = \dfrac{-2\mathcal{H}}{\pi \kappa \sigma (p-2)(p-3)} > 0.$

Proof:
The proof uses analogous steps as in the last one and can be omitted. For the sake of completeness we perform it in the following.
Here we have with $c_0 = 2\pi(2-p)^{-1}(3-p)^{-1}$, $c_1 = 2\pi(2-p)^{-1}$ and $c_2 = 2\pi$

$$
\begin{aligned}
g(x_1) &= c_0 \left\{ (d+x_1)^{3-p} + (d-x_1)^{3-p} \right\} \\
g'(x_1) &= c_1 \left\{ (d+x_1)^{2-p} - (d-x_1)^{2-p} \right\} \\
g''(x_1) &= c_2 \left\{ (d+x_1)^{1-p} + (d-x_1)^{1-p} \right\}
\end{aligned}
$$

where g is up to a constant factor the disjoining pressure potential. So we can transform $\operatorname{div} \mathbf{T}\, g$

$$
\begin{aligned}
\operatorname{div} \mathbf{T}\, g &= g'' \left(1+g'^2\right)^{-\frac{3}{2}} \\
&= c_2 \left[\chi_+^{1-p} + \chi_-^{1-p} \right] \cdot \left[1 + c_1^2 \left\{ \chi_+^{2-p} - \chi_-^{2-p} \right\}^2 \right]^{-\frac{3}{2}} \\
&= c_2 \chi_+^{1-p} \left[1 + \left(\frac{\chi_+}{\chi_-} \right)^{p-1} \right] \cdot \left[\left(\chi_+^{2p-4} + c_1^2 \left\{ 1 - \left(\frac{\chi_+}{\chi_1} \right)^{p-2} \right\}^2 \right) \chi_+^{4-2p} \right]^{-\frac{3}{2}} \\
&= c_2 \chi_+^{2p-5} \left[1 + \left(\frac{\chi_+}{\chi_-} \right)^{p-1} \right] \cdot \left[\chi_+^{2p-4} + c_1^2 \left\{ 1 - \left(\frac{\chi_+}{\chi_-} \right)^{p-2} \right\}^2 \right]^{\frac{3}{2}} ,
\end{aligned}
\tag{3.10}
$$

where we have set $\chi_\pm = d \pm x_1$. Analogously we get

$$\operatorname{div} \mathbf{T}\, g = c_2 \chi_-^{2p-5} \left[\left(\frac{\chi_-}{\chi_+} \right)^{p-1} + 1 \right] \cdot \left[\chi_-^{2p-4} + c_1^2 \left\{ \left(\frac{\chi_-}{\chi_+} \right)^{p-2} - 1 \right\}^2 \right]^{-\frac{3}{2}} . \tag{3.11}$$

Taking into consideration, that $2p - 5 > 0$ it follows from the last part of formula (3.10) and from (3.11) that $\operatorname{div} \mathbf{T}\, g$ tends to 0 as x_1 tends to $\pm d$.
So the theorem is proven. □

3.6 Proof of Theorem 11

Recall that

$$\Omega = \left\{ x \in \mathbb{R}^2 : |x_2| < x_1 \tan \alpha \right\}$$

and

$$\Omega_\rho = \left\{ x \in \mathbb{R}^2 : |x_2| < x_1 \tan \alpha \right\} \cap B_\rho$$

where B_ρ denotes a ball of radius ρ with $0 < \rho < 1$. In addition we define $\Gamma_\rho = \Omega \cap \partial B_\rho$ and $\Sigma = \partial\Omega \cap B_\rho$.

Theorem 11
Let v be a solution of (2.2) for $p = 6$ over the domain Ω, $0 < \alpha < pi/2$. Then there are constants $0 < \rho < 1$, $B \in \mathbb{R}$ and $A \geq 0$, independent of the special solution v considered, so that we have

$$\left| v(r,\theta) - \frac{f(\theta)}{r^3} - \frac{h(\theta)}{r} \right| \leq A \qquad \text{in } \Omega,$$

and

$$\left| v(r,\theta) - \frac{f(\theta)}{r^3} - \frac{h(\theta)}{r} - B \right| \leq A \cdot r \qquad \text{in } \Omega_\rho.$$

Thereby f and h are given by

$$f(\theta) = C \cdot \left(3 \cot \frac{\alpha+\theta}{2} + 3 \cot \frac{\alpha-\theta}{2} + \cot^3 \frac{\alpha+\theta}{2} + \cot^3 \frac{\alpha-\theta}{2} \right), \qquad C = \frac{-\mathcal{H}}{16\pi\sigma\kappa} > 0,$$
$$h(\theta) = \frac{3}{\kappa} f(\theta) \frac{3f(\theta) \cdot f''(\theta) - 9f(\theta)^2 - 4f'(\theta)^2}{[9f(\theta)^2 + f'(\theta)^2]^{3/2}}.$$

We divide the proof of Theorem 11 into five parts. These parts verify the assumptions for Corollary 12 which then yields the first statement. When $|v - h/r^3 - h/r| < B < \infty$ in Ω is known, we can verify condition (ii) of Corollary 13 and apply it to get the desired second assertion in Ω_ρ.

Corollary 12
Let v be a solution of (2.2) over the domain Ω and w a suitable comparison function. Then we have:

a) If

$$\begin{array}{lrl} \text{(i)} & \operatorname{div} \boldsymbol{T}w - \kappa w - P(x,w) \leq 0 & \textit{in } \Omega \\ \text{(ii)} & \nu \cdot \boldsymbol{T}w \geq 1 & \textit{on } \Sigma \setminus \{0\}, \end{array}$$

then we have $w \geq v$ in Ω.
b) If

$$\begin{array}{lrl} \text{(i)} & \operatorname{div} \boldsymbol{T}w - \kappa w - P(x,w) \geq 0 & \textit{in } \Omega \\ \text{(ii)} & \nu \cdot \boldsymbol{T}w \leq 1 & \textit{on } \Sigma \setminus \{0\}, \end{array}$$

then we have $w \leq v$ in Ω.

Corollary 13
Let v be a solution of (2.2) over the domain Ω_ρ, w a suitable comparison function and $0 < \rho < 1$ sufficiently small. Then we have:
a) If

$$\begin{array}{lrl} \text{(i)} & \operatorname{div} \boldsymbol{T}w - \kappa w - P(x,w) \leq 0 & \textit{in } \Omega_\rho \\ \text{(ii)} & w \geq v & \textit{on } \Gamma_\rho \\ \text{(iii)} & \nu \cdot \boldsymbol{T}w \geq 1 & \textit{on } \Sigma_\rho \setminus \{0\}, \end{array}$$

then we have $w \geq v$ in Ω_ρ. b) If

$$\begin{array}{lrl} \text{(i)} & \operatorname{div} \boldsymbol{T}w - \kappa w - P(x,w) \geq 0 & \textit{in } \Omega_\rho \\ \text{(ii)} & w \leq v & \textit{on } \Gamma_\rho \\ \text{(iii)} & \nu \cdot \boldsymbol{T}w \leq 1 & \textit{on } \Sigma_\rho \setminus \{0\}, \end{array}$$

then we have $w \leq v$ in Ω_ρ.

Proof:

3.6.1 Disjoining pressure potential

The first step is to compute the disjoining pressure potential. The second step is to use it to find a suitable comparison function w to apply Corollary 12 and Corollary 13 respectively. Due to the present symmetry, it is clear that we are losing the rotational symmetry in this case. To obtain an asymptotic result in the cusp, we have the domain $\Omega_\rho = \Omega \cap B_\rho$. Define $\Sigma_\rho = \partial\Omega \cap B_\rho$ and $\Gamma_\rho = \partial B_\rho \cap \Omega$, see Figure 3.5. Actually, after converting problem (2.2) in cylindrical coordinates, as we did in a previous section, we get the following for the disjoining pressure potential $K = K(r,\theta) = P(x,u(x))$

$$\begin{aligned} K &= c \int_\alpha^{2\pi-\alpha} \int_0^\infty \int_{-\infty}^\infty s \left[(r\cos\theta - s\cos\varphi)^2 + (r\sin\theta - s\sin\varphi)^2 + (U-h)^2 \right]^{-p/2} \mathrm{d}h \mathrm{d}s \mathrm{d}\varphi \\ &= 2c \int_\alpha^{2\pi-\alpha} \int_0^\infty \int_0^\infty s \left[s^2 - 2rs\cos(\varphi-\theta) + r^2 + h^2 \right]^{-p/2} \mathrm{d}h \mathrm{d}s \mathrm{d}\varphi. \end{aligned}$$

At this point we use the transformation $h = \tilde{c}\tan t$, $0 < t < \pi/2$ and $\mathrm{d}h = \tilde{c}(\cos t)^{-2}\mathrm{d}t$, whereby $\tilde{c}^2 = s^2 - 2rs\cos(\varphi-\theta) + r^2$. With this transformation and using

$$\int_0^{\pi/2} (\cos t)^{p-2} \mathrm{d}t = 2^{p-3} \mathrm{B}\left(\frac{p-1}{2}, \frac{p-1}{2}\right) = 2^{p-3} \frac{\Gamma^2\left(\frac{p-1}{2}\right)}{\Gamma(p-1)} = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{p-1}{2}\right)}{\Gamma(p/2)},$$

see [GR63, 3.421.2] and [AS64, 6.2.2, 6.1.18] we obtain

$$
\begin{aligned}
K &= 2c\int_{\alpha}^{2\pi-\alpha}\int_0^{\infty} s\left[s^2-2rs\cos(\varphi-\theta)+r^2\right]^{(1-p)/2}\mathrm{d}s\mathrm{d}\varphi\int_0^{\frac{\pi}{2}}(\cos t)^{p-2}\mathrm{d}t\\
&=\sqrt{\pi}\frac{\Gamma\left(\frac{p-1}{2}\right)}{\Gamma(p/2)}c\int_{\alpha}^{2\pi-\alpha}\int_0^{\infty} s\left[s^2-2rs\cos(\varphi-\theta)+r^2\right]^{(1-p)/2}\mathrm{d}s\mathrm{d}\varphi\\
&=\sqrt{\pi}\frac{\Gamma\left(\frac{p-1}{2}\right)}{\Gamma(p/2)}c\int_{\alpha-\theta}^{2\pi-\alpha-\theta}\int_0^{\infty} s\left[s^2-2rs\cos\varphi+r^2\right]^{(1-p)/2}\mathrm{d}s\mathrm{d}\varphi.
\end{aligned}
$$

In general the integration with respect to s implies the associated Legendre functions, see

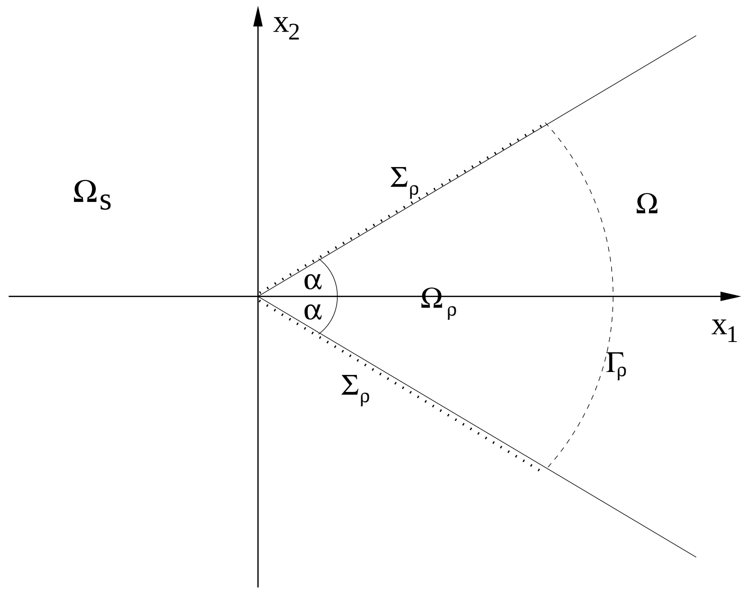

Figure 3.5: Wedge-shaped domain.

[GR82, Eq. 3.252.10]. For the important physical case $p = 6$ we obtain some explicit results. So from now on, let be $p = 6$. Then, considering $\Gamma(5/2) = 3\sqrt{\pi}/4$ and using Formula [GR82, Eq. 3.252.7]:

$$
\int_0^{\infty}\frac{x^n\mathrm{d}x}{(ax^2+2bx+c)^{n+3/2}}=\frac{n!}{(2n+1)!!\sqrt{c}(\sqrt{ac}+b)^{n+1}},
$$

(with $n = 1$) and [GR63, Eq. 2.453.7],

$$
\int\frac{\mathrm{d}x}{(1-\cos x)^n}=-2^{1-n}\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\cot^{2k+1}\frac{x}{2}}{2k+1}
$$

one gets

$$\begin{aligned} K &= \frac{3\pi}{8} c \int_{\alpha-\theta}^{2\pi-\alpha-\theta} \int_0^\infty \frac{s^1 \mathrm{d}s}{[s^2 - 2rs\cos\varphi + r^2]^{1+3/2}} \mathrm{d}\varphi \\ &= \frac{\pi}{8} c r^{-3} \int_{\alpha-\theta}^{2\pi-\alpha-\theta} \frac{\mathrm{d}\varphi}{(1-\cos\varphi)^2} \\ &= \frac{\pi}{48} c r^{-3} \left\{ 3\cot\frac{\alpha+\theta}{2} + 3\cot\frac{\alpha-\theta}{2} + \cot^3\frac{\alpha+\theta}{2} + \cot^3\frac{\alpha-\theta}{2} \right\}. \end{aligned}$$

So we obtain $P(x, u(x)) = \tilde{c}\psi(\theta) r^{-3}$, where $\tilde{c} = \mathcal{H}/(48\sigma\pi) < 0$ and

$$\psi(\theta) = 3\cot\frac{\alpha+\theta}{2} + 3\cot\frac{\alpha-\theta}{2} + \cot^3\frac{\alpha+\theta}{2} + \cot^3\frac{\alpha-\theta}{2}. \tag{3.12}$$

We state some properties of ψ that will help us to perform some calculations in the following proof. The function ψ has got the derivatives

$$\begin{aligned} \psi'(\theta) &= \frac{3}{2}\left(\sin\frac{\alpha-\theta}{2}\right)^{-4} - \frac{3}{2}\left(\sin\frac{\alpha+\theta}{2}\right)^{-4} \qquad \text{and} \\ \psi''(\theta) &= 3\cos\frac{\alpha-\theta}{2}\left(\sin\frac{\alpha-\theta}{2}\right)^{-5} + 3\cos\frac{\alpha+\theta}{2}\left(\sin\frac{\alpha+\theta}{2}\right)^{-5}. \end{aligned} \tag{3.13}$$

The function $\cot x$ has got the following Taylor series

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n} B_n}{(2n)!} x^{2n-1}, \text{ for } 0 < |x| < \pi,$$

where B_n is the n-th Bernoulli number. That is, we have

$$\cot x = \frac{1}{x} + O(x), \text{ as } x \to 0.$$

So we deduce from equation (3.12) and (3.13) respectively[2]

$$\begin{aligned} \psi(\theta) &\sim 8(\alpha \pm \theta)^{-3}, \\ \psi'(\theta) &\sim \mp 24(\alpha \pm \theta)^{-4}, \\ \psi''(\theta) &\sim 96(\alpha \pm \theta)^{-5}, \end{aligned} \tag{3.14}$$

as θ tends to $\mp\alpha$.
From (3.12) and (3.13) we can conclude that ψ, ψ' and ψ'' are continuous and bounded in $(-\alpha + \varepsilon_0, \alpha + \varepsilon_0)$ for a sufficiently small, positive ε_0.

3.6.2 Requirements in Ω_ρ and Ω

Our next aim is to find an appropriate comparison function to estimate $\operatorname{div} \mathbf{T}\, u - \kappa u - P(x)$. In the previous sections the divergence of $P(x, u(x))$ was bounded. As we will see in the case of the wedge, the divergence of $\mathbf{T}\, P$ is asymptotically equal to $\kappa h(\theta) \cdot r^{-1}$ as r tends to 0 for

[2] For the notation of $f \sim g$, see page 12.

some non-vanishing function h. So we have to find another suitable comparison function. It turns out that the following function $w(\theta, r)$ is an adequate choice.
So consider the comparison function

$$w(\theta, r) = \frac{f(\theta)}{r^3} + \frac{h(\theta)}{r} + A, \tag{3.15}$$

with

$$f = f(\theta) = -\frac{\tilde{c}}{\kappa}\psi(\theta), \qquad h = h(\theta) = \frac{3}{\kappa} f(\theta) \frac{3f(\theta) \cdot f''(\theta) - 9f(\theta)^2 - 4f'(\theta)^2}{[9f(\theta)^2 + f'(\theta)^2]^{3/2}}, \tag{3.16}$$

where A and $\tilde{c} = \mathcal{H}/(48\sigma\pi) < 0$ are constants. So we get (see Lemma 14) $h \in \mathcal{C}^2([-\alpha, \alpha])$ and in addition

$$\begin{aligned} h(\theta) &= O([\alpha \pm \theta]^3), \text{ as } \theta \to \mp\alpha, \\ h'(\theta) &= O([\alpha \pm \theta]^2), \text{ as } \theta \to \mp\alpha, \\ h''(\theta) &= O([\alpha \pm \theta]^1), \text{ as } \theta \to \mp\alpha. \end{aligned}$$

Step 1:
In this step we consider the behaviour of $\operatorname{div} \mathbf{T}\, w$ in Ω_ρ, for a ρ with $0 < \rho < 1$. That is, we observe the behaviour near the cusp.
From (3.15) we obtain the characteristics

$$\begin{aligned} w_r &= -r^{-4}(3f + hr^2), \\ w_\theta &= r^{-3}(f' + h'r^2), \\ rw_r &= -r^{-4}(3fr + hr^3), \\ r^{-1}w_\theta &= r^{-4}(f' + h'r^2), \end{aligned}$$

and therewith

$$\begin{aligned} |\nabla w|^2 = w_r^2 + r^{-2}w_\theta^2 &= r^{-8}\left\{(3f + hr^2)^2 + (f' + h'r^2)^2\right\} \\ &= r^{-8}\left\{9f^2 + f'^2 + 2(3fh + f'h')r^2 + (h^2 + h'^2)r^4\right\} \\ &= r^{-8}\left\{9f^2 + f'^2 + Q\right\}, \end{aligned}$$

where we have set

$$Q = 2(3fh + f'h')r^2 + (h^2 + h'^2)r^4 = 2(3fh + f'h')r^2 + O(r^4) = O([1 + [\alpha \pm \theta]^{-2}]r^2). \tag{3.17}$$

So we obtain the partial derivatives of Q

$$Q_r = 4(3fh + f'h')r + 4(h^2 + h'^2)r^3 = O([1 + [\alpha \pm \theta]^{-2}]r) \tag{3.18}$$

and

$$Q_\theta = 2(3fh + f'h')'r^2 + (h^2 + h'^2)'r^4 = O([1 + [\alpha \pm \theta]^{-3}]r^2).$$

Finally, for sufficiently small ρ, $0 < r < \rho < 1$ and $a \in \mathbb{R}$, we bring the following term in a convenient form

$$\begin{aligned}\left[r^8 + 9f^2 + f'^2 + Q\right]^a &= \left(9f^2 + f'^2\right)^a \left(1 + \frac{Q + r^8}{9f^2 + f'^2}\right)^a \\ &= \left(9f^2 + f'^2\right)^a \left[1 + 2a\frac{3fh + f'h'}{9f^2 + f^2} r^2 + O(r^4)\right] \\ &= \left(9f^2 + f'^2\right)^a \left[1 + O(r^2)\right].\end{aligned} \tag{3.19}$$

By definition of Q and h, the term $Q/(9f^2 + f'^2)$ is bounded, $9f^2 + f'^2$ is positive. So these transformations are valid.
Recall that the divergence of $\mathbf{T}\, w$ in polar coordinates is given as:

$$\operatorname{div} \mathbf{T}\, w = \frac{1}{r}\left\{ \left(\frac{r w_r}{\sqrt{1 + |\nabla w|^2}}\right)_r + \left(\frac{r^{-1} w_\theta}{\sqrt{1 + |\nabla w|^2}}\right)_\theta \right\}.$$

Now we start to compute $\operatorname{div} \mathbf{T}\, w$ beginning with the term:

$$\begin{aligned}\left(\frac{r w_r}{\sqrt{1 + |\nabla w|^2}}\right)_r &= \left(\frac{-3fr - hr^3}{\sqrt{r^8 + 9f^2 + f'^2 + Q}}\right)_r \\ &= \frac{[-3f - 3hr^2]\sqrt{r^8 + 9f^2 + f'^2 + Q} - \frac{1}{2}[-3fr - hr^3]\frac{8r^7 + Q_r}{\sqrt{r^8 + 9f^2 + f'^2 + Q}}}{r^8 + 9f^2 + f'^2 + Q}\end{aligned}$$

At this point we use the boundedness of h and h' as well as the asymptotic behaviour of f and the relations (3.17), (3.18) and (3.19) to compute

$$\begin{aligned}\left(\frac{r w_r}{\sqrt{1 + |\nabla w|^2}}\right)_r &= \frac{-3f + O(r^2)}{\sqrt{r^8 + 9f^2 + f'^2 + Q}} + \frac{1}{2}\frac{[3fr + O(r^3)] \cdot O([1 + [\alpha \pm \theta]^{-2}]r)}{[r^8 + 9f^2 + f'^2 + Q]^{\frac{3}{2}}} \\ &= \frac{-3f + O(r^2)}{\sqrt{9f^2 + f'^2}}[1 + O(r^2)] + \frac{1}{2}\frac{O([1 + [\alpha \pm \theta]^{-5}]r^2)}{[r^8 + 9f^2 + f'^2 + Q]^{\frac{3}{2}}} \\ &= \frac{-3f + O(r^2)}{\sqrt{9f^2 + f'^2}} + O(r^2) + \frac{O([1 + [\alpha \pm \theta]^{-5}]r^2)}{[9f^2 + f'^2]^{\frac{3}{2}}}[1 + O(r^2)] \\ &= \frac{-3f}{\sqrt{9f^2 + f'^2}} + O(r^2).\end{aligned}$$

To obtain the term $O(r^2)$ we use the continuity of f and f' in $(-\alpha + \varepsilon_0, \alpha - \varepsilon_0)$ and their asymptotic characteristics (3.14). In a similar way we treat the next term:

$$\begin{aligned}\left(\frac{r^{-1}w_\theta}{\sqrt{1+|\nabla w|^2}}\right)_\theta &= \left(\frac{f'+h'r^2}{\sqrt{r^8+9f^2+f'^2+Q}}\right)_\theta \\ &= \frac{(f''+h''r^2)\sqrt{r^8+9f^2+f'^2+Q}-\frac{1}{2}(f'+h'r^2)\frac{2f'(9f+f'')+Q_\theta}{\sqrt{r^8+9f^2+f'^2+Q}}}{r^8+9f^2+f'^2+Q} \\ &= \frac{[f''+O(r^2)][9f^2+f'^2+O([1+[\alpha\pm\theta]^{-2}r^2)]}{[r^8+9f^2+f'^2+Q]^{\frac{3}{2}}} \\ &\quad -\frac{(f'+h'r^2)[f'(9f+f'')+O([1+(\alpha\pm\theta)^{-3}]r^2)]}{[r^8+9f^2+f'^2+Q]^{\frac{3}{2}}} \\ &= \frac{9f(ff''-f'^2)+O([1+[\alpha\pm\theta]^{-8}]r^2)}{[r^8+9f^2+f'^2+Q]^{\frac{3}{2}}} \\ &= 9f\frac{ff''-f'^2}{[9f^2+f'^2]^{\frac{3}{2}}}+O(r^2).\end{aligned}$$

So we are able calculate $\operatorname{div}\mathbf{T}\,w$:

$$\operatorname{div}\mathbf{T}\,w = \frac{3f}{r}\cdot\frac{3ff''-9f^2-4f'^2}{[9f^2+f'^2]^{3/2}}+O(r) = \frac{\kappa}{r}h+O(r). \tag{3.20}$$

To adopt the Comparison Principle and the corresponding corollaries respectively we derive the following formula, by using formulae (3.15) and (3.20) and the definition of P

$$\begin{aligned}\operatorname{div}\mathbf{T}\,w-\kappa w-P &= \kappa hr^{-1}+O(r)-\kappa fr^{-3}-\kappa hr^{-1}-\kappa A+\kappa fr^{-3} \\ &= -\kappa A+O(r).\end{aligned}$$

Then we can choose the constant A positive so that we have

$$\operatorname{div}\mathbf{T}\,w-\kappa w-P\leq 0. \tag{3.21}$$

Otherwise, by choosing A negative, we can achieve $\operatorname{div}\mathbf{T}\,w-\kappa w-P\geq 0$.

Step 2:
Here we regard $\operatorname{div}\mathbf{T}\,w-\kappa w-P$ in Ω. The aim is to show that $|\operatorname{div}\mathbf{T}\,w-\kappa w-P|<\infty$. From the first step one easily concludes this relation in Ω_ρ for $0<\rho<1$. So from now on we consider $r\geq\rho>0$. The "c" which occurs in the next calculations denotes some positive constant. When it appears repeatedly in a calculation, it does not always need to represent the same value. The important thing is that it is a bounded quantity, the actual value is of secondary importance.

From the following calculation it is easy to see that $\operatorname{div}\mathbf{T}\,w$ is bounded when $r>0$ is finite and θ is arbitrary or if $\theta\in(-\alpha+\varepsilon_0,\alpha-\varepsilon_0)$ and r is arbitrary, for some small positive ε_0. Consequently it is sufficient to examine the case that θ tends to $\pm\alpha$ and r tends to infinity. So we can choose $\varepsilon_0>0$ small enough and r big enough that

$$r^8+9f^2+f'^2+(3fh+f'h')r^2+(h^2+h'^2)r^4\geq r^8+f'^2>f'^2>1 \tag{3.22}$$

is satisfied. Remember that $3fh + f'h'$ becomes positive as θ tends to $\pm\alpha$, see Lemma 14. Under this condition we can compute $|\mathrm{div}\, \mathbf{T}\, w|$, beginning with:

$$\begin{aligned}\left|\left(\frac{rw_r}{\sqrt{1+|\nabla w|^2}}\right)_r\right| &= \left|\left(\frac{-3fr - hr^3}{\sqrt{r^8+9f^2+f'^2+Q}}\right)_r\right| \\ &\leq \left|\frac{3f+3hr^2}{r^4\sqrt{1+|\nabla w|^2}}\right| + \left|\frac{1}{2}\frac{(3fr+hr^3)(8r^7+Q_r)}{r^{12}\left[1+|\nabla w|^2\right]^{\frac{3}{2}}}\right| \\ &\leq c\frac{|w_r|}{\sqrt{1+|\nabla w|^2}} \\ &\quad + \left|\frac{3fr^{-4}+hr^{-2}}{\sqrt{1+|\nabla w|^2}} \cdot \frac{4r^8+2(3fh+f'h')r^2+2(h^2+h'^2)r^4}{r^8\left[1+|\nabla w|^2\right]}\right|.\end{aligned}$$

Now, considering that fh, h, h' are bounded, we can further estimate

$$\begin{aligned}\left|\left(\frac{rw_r}{\sqrt{1+|\nabla w|^2}}\right)_r\right| &\leq c\frac{|\nabla w|}{\sqrt{1+|\nabla w|^2}} + \frac{|w_r|}{\sqrt{1+|\nabla w|^2}} \cdot \left|\frac{cr^8+f'h'r^2}{r^8\left[1+|\nabla w|^2\right]}\right| \\ &\leq c + c\left|\frac{cr^8+f'h'r^2}{r^8\left[1+|\nabla w|^2\right]}\right| \leq c + \frac{c}{1+|\nabla w|^2} + c\left|\frac{f'h'r^2}{r^8\left[1+|\nabla w|^2\right]}\right| \\ &\overset{(3.22)}{\leq} c + c\left|\frac{f'h'r^2}{r^8+f'^2}\right| \leq c + c\left|\frac{f'h'r^2}{r^4 f'}\right| = c + c\left|h'r^{-2}\right| \leq c.\end{aligned} \tag{3.23}$$

For the last step we used the inequality of arithmetic and geometric means that is, $r^8 + f'^2 \geq cr^4|f'|$.

Now we compute

$$\begin{aligned}\left|\left(\frac{r^{-1}w_\theta}{\sqrt{1+|\nabla w|^2}}\right)_\theta\right| &= \left|\left(\frac{f'+h'r^2}{\sqrt{r^8+9f^2+f'^2+Q}}\right)_\theta\right| \\ &= \left|\frac{(f''+h''r^2)\sqrt{r^8+9f^2+f'^2+Q} - \frac{1}{2}(f'+h'r^2)\dfrac{2f'(9f+f'')+Q_\theta}{\sqrt{r^8+9f^2+f'^2+Q}}}{r^8+9f^2+f'^2+Q}\right| \\ &= \left|\frac{9f(ff''-f'^2)+3\left[3f^2h''-2f(2f'h'-f''h)-f'^2h\right]r^2}{\left[r^8+9f^2+f'^2+Q\right]^{3/2}}\right. \\ &\quad \left. + \frac{\left[3f(2hh''-h'^2)-4f'hh'+f''h^2\right]r^4+\left[h(hh''-h'^2)\right]r^6+f''r^8+h''r^{10}}{\left[r^8+9f^2+f'^2+Q\right]^{3/2}}\right|.\end{aligned}$$

Therefore by using the asymptotic behaviour of f, h and their derivatives, we can estimate the upper expression for some positive constant c, by three terms:

$$\begin{aligned}
&\frac{c\cdot(\alpha\pm\theta)^{-11}}{[r^8+9f^2+f'^2+Q]^{3/2}}+\frac{c\cdot r^{10}}{[r^8+9f^2+f'^2+Q]^{3/2}}+\frac{c\cdot(\alpha\pm\theta)^{-5}r^8}{[r^8+9f^2+f'^2+Q]^{3/2}}\\
&\quad\le c\frac{(\alpha\pm\theta)^{-11}}{[f'^2]^{3/2}}+\frac{c\cdot r^{10}}{[r^8]^{3/2}}+c\,r\frac{(\alpha\pm\theta)^{-5}}{[r^8+f'^2]^{5/8}}\cdot\frac{r^7}{[r^8+f'^2]^{7/8}}\\
&\quad\le c+c\,r\frac{(\alpha\pm\theta)^{-5}}{[f'^2]^{5/8}}\cdot\frac{r^7}{[r^8]^{7/8}}\le c\cdot r,
\end{aligned}\tag{3.24}$$

where we have used (3.22). Therefore by combining (3.23), (3.24) and the considerations at the beginning of the step we get

$$|\mathrm{div}\,\mathbf{T}\,w|=r^{-1}\left|\left(\frac{rw_r}{\sqrt{1+|\nabla w|^2}}\right)_r+\left(\frac{r^{-1}w_\theta}{\sqrt{1+|\nabla w|^2}}\right)_\theta\right|\le c,$$

in $\Omega\setminus\Omega_\rho$. Therefore by choosing A appropriate positive we can estimate in $\Omega\setminus\Omega_\rho$

$$\mathrm{div}\,\mathbf{T}\,w-\kappa w-P\le c-\kappa\,hr^{-1}-\kappa A\le 0,\tag{3.25}$$

since h is bounded and $r>\rho>0$.
Otherwise, by choosing A negative, we can achieve $\mathrm{div}\,\mathbf{T}\,w-\kappa w-P\ge 0$.

3.6.3 Boundary behaviour

To prove the boundary condition we have to show

$$\nu\cdot\mathbf{T}\,w=\mathrm{sign}(\theta)\frac{r^{-1}w_\theta}{\sqrt{1+|\nabla w|^2}}=1,\text{ as }\theta\to\pm\alpha$$

for every $r>0$. To show this we consider at first the case $0<r<\rho<1$ and $\theta\in(-\alpha,-\alpha+\varepsilon_0)\cup(\alpha-\varepsilon_0,\alpha)$, for a sufficiently small $\varepsilon_0>0$. Therefore we can use the asymptotic behaviour of the occurring functions. In detail, we have

$$\begin{aligned}
\frac{r^{-1}w_\theta}{\sqrt{1+|\nabla w|^2}}&=\frac{f'+h'r^2}{\sqrt{r^8+9f^2+f'^2+Q}}=\frac{f'+h'r^2}{\sqrt{9f^2+f'^2}}\left[1+\frac{r^8+Q}{9f^2+f'^2}\right]^{-1/2}\\
&=\frac{f'}{\sqrt{9f^2+f'^2}}\left(1+\frac{h'}{f'}r^2\right)\left[1+\frac{r^8+2(3fh+f'h')r^2+(h^2+h'^2)r^4}{9f^2+f'^2}\right]^{-1/2}\\
&=\frac{f'}{\sqrt{9f^2+f'^2}}\left(1+\frac{h'}{f'}r^2\right)\left[1+O\left(\frac{f'h'}{9f^2+f'^2}r^2\right)\right]\\
&=\frac{f'}{\sqrt{9f^2+f'^2}}\left[1+O\left(\frac{f'h'}{9f^2+f'^2}r^2\right)\right]\\
&=\frac{f'}{\sqrt{9f^2+f'^2}}+O\left(\frac{f'^2h'}{[9f^2+f'^2]^{3/2}}r^2\right).
\end{aligned}$$

Since $f\sim c_0(\alpha\pm\theta)^{-3}$ and $f'\sim c_1(\alpha\pm\theta)^{-4}$, with $c_0\cdot c_1\ne 0$, the last expression tends to ±1, as θ tends to $\pm\alpha$. The other transformations are valid since h and h' are bounded (see Lemma 14).

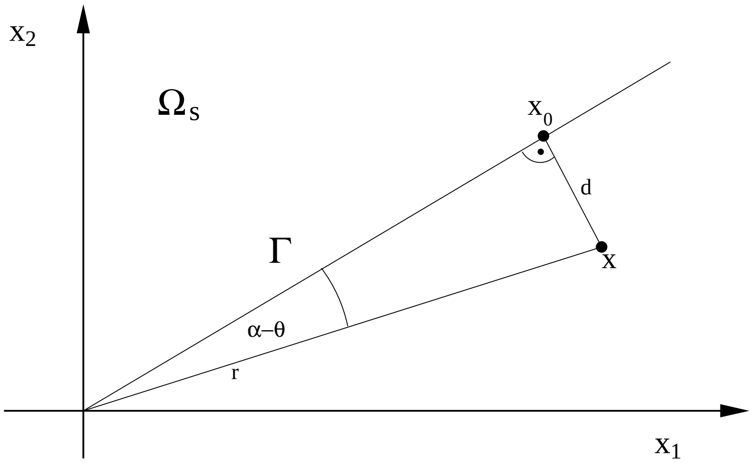

Figure 3.6: Normal distance to Σ.

Let $x \in \Omega$, $x = r(\cos\theta, \sin\theta)$ for $r > \rho > 0$, $\theta \in (0, \alpha)$. Hence the normal distance of x to Σ is $d = r\sin(\alpha - \theta)$, see Figure 3.6. To check the behaviour of $\nu \cdot \mathbf{T}\, w$ on Σ afar from the origin, the normal distance of x to Σ has to tend to 0. That is, we have in particular $|\alpha - \theta| << 1$ and therefore $d = r(\alpha - \theta) + O([r(\alpha - \theta)]^3)$.
Thus we have

$$\begin{aligned} r^2 h'/f' &\sim c_1(\alpha-\theta)[r(\alpha-\theta)]^2, \\ r^8/f'^2 &\sim c_2[r(\alpha-\theta)]^8, \\ 9f^2/f'^2 &\sim c_3(\alpha-\theta)^2, \\ 2(3fh + f'h')r^2/f'^2 &\sim c_4(\alpha-\theta)^4[r(\alpha-\theta)]^2, \\ (h^2 + h'^2)r^4/f'^2 &\sim c_5(\alpha-\theta)^8[r(\alpha-\theta)]^4, \end{aligned} \tag{3.26}$$

whereby $c_0, \ldots, c_6$ are some bounded constants.
Combining the definition of $\nu \cdot \mathbf{T}\, w$ with (3.26) we get

$$\frac{r^{-1}w_\theta}{\sqrt{1 + |\nabla w|^2}} = \frac{1 + \dfrac{h'}{f'}r^2}{\sqrt{1 + r^8/f'^2 + 9f^2/f'^2 + \dfrac{2(3fh + f'h')r^2 + (h^2 + h'^2)r^4}{f'^2}}},$$

which tends to 1, as x approaches Σ, since all expressions of (3.26) tend to 0, as $r\sin(\alpha - \theta)$ tends to 0.
And so the boundary condition is shown.

3.6.4 Applying the Comparison Principle

Let v be a solution of (2.2) and $w = fr^{-3} + hr^{-1} + A$. Firstly we apply the Comparison Principle to show that v satisfies $|v - fr^{-3} - hr^{-1}| \leq A$ in Ω for some positive A. And therewith we show $|v - fr^{-3} - hr^{-1} - B| < A \cdot r$ in Ω_ρ, for a ρ with $0 < \rho < 1$.
After choosing $A > 0$ sufficiently big we can combine (3.21) and (3.25) to get

$$\operatorname{div} \mathbf{T}\, w - \kappa w - P \leq 0 \text{ in } \Omega.$$

Therefore condition (i) of Corollary 12.a) is satisfied. Since we have shown that

$$\nu \cdot \mathbf{T}\, w = 1 \text{ as } x \to \Sigma \setminus \{0\},$$

condition (ii) of Corollary 12.a) is satisfied. So we get $v \leq w$ in Ω. Similarly by choosing A negative we get $v \geq w$. Therewith we have

$$\left| v(r,\theta) - \frac{f(\theta)}{r^3} - \frac{h(\theta)}{r} \right| \leq A \text{ in } \Omega,$$

for some constant $A > 0$.
From now on we consider just the cusp of the domain Ω that is Ω_ρ, where $0 < \rho < 1$. Replacing the upper comparison function w by $\tilde{w} = fr^{-3} + hr^{-1} + B + Ar$ results just a slight modification[3] of the upper calculations. Thereby f and h are the functions defined in (3.16) and A and B constants. Finally one also attains

$$\operatorname{div} \mathbf{T}\, \tilde{w} = \kappa h r^{-1} + O(r) \qquad \text{in } \Omega_\rho.$$

Thus we can choose the constant $A > 0$ appropriate big to get

$$\operatorname{div} \mathbf{T}\, \tilde{w} - \kappa \tilde{w} - P = -\kappa A r + O(r) \leq 0,$$

and again, by choosing A negative, the vice versa relation can be achieved. In other words property (i) of Corollary 13 is satisfied.
Thus again we have $\nu \cdot \operatorname{div} \mathbf{T}\, \tilde{w} = 1$ as $x \to \Sigma_\rho \setminus \{0\}$, which is property (iii) of Corollary 13. The last step is to show property (ii) of the same corollary. By choosing $|B|$ sufficiently large we can achieve this since we have shown that $|v - fr^{-3} - hr^{-1}| \leq B$, where v is the solution of (2.2).
Then we can apply Corollary 13 to achieve $|v - fr^{-3} - hr^{-1} - B| \leq Ar$, which means the second part of the theorem is shown, which proves the theorem. □

The result for $\alpha \in (\pi/2, \pi)$ follows from the calculations above.

3.6.5 Properties of h

Lemma 14
Let h be the function defined by equation (3.16). Then we have:

a) $h \in \mathcal{C}^2([-\alpha, \alpha])$,

b) $h^{(k)} = O([\alpha \pm \theta]^{3-k})$, *as θ tends to $\mp\alpha$ and $k = 0, 1, 2$,*

c) h, h' *and* h'' *are bounded,*

d) $f' \cdot h' > 0$ *and* $3fh + f'h' > 0$, *as θ tends to $\pm\alpha$.*

[3] The additonal term Ar has no perturbing influence on the calculations since it is bounded by f/r^3 and h/r, respectively.

Proof:
We have got

$$h = \frac{3}{\kappa} f \frac{3ff'' - 9f^2 - 4f'^2}{[9f^2 + f'^2]^{3/2}} \tag{3.27}$$

and we get for the derivatives

$$h' = \frac{h_1}{[9f^2 + f'^2]^{5/2}} \qquad \text{and} \qquad h'' = \frac{h_2}{[9f^2 + f'^2]^{7/2}},$$

where h_1 is a homogeneous polynomial of fifth degree in f, f', f'' and f''' and h_2 a homogeneous polynomial of seventh degree in f, f', f'', f''' and $f^{(4)}$.
Since f is a multiple of ψ, it is sufficient to examine ψ and its derivatives to verify the stated properties. From the equations (3.12) and (3.13) we see that ψ, ψ' and ψ'' are continuous in $(-\alpha+\varepsilon_0, \alpha-\varepsilon_0)$ for a sufficiently small $\varepsilon_0 > 0$, and from (3.14) we have got their asymptotic behaviour as θ tends to $\pm\alpha$. After calculating ψ''' and $\psi^{(4)}$, one obtains that they also are continuous in $(-\alpha+\varepsilon_0, \alpha-\varepsilon_0)$ and have a similar asymptotic behaviour as $\theta \to \mp\alpha$. More precisely we have got

$$\begin{gathered} \psi \sim 8(\alpha \pm \theta)^{-3}, \qquad \psi' \sim \mp 24(\alpha \pm \theta)^{-4}, \qquad \psi'' \sim 96(\alpha \pm \theta)^{-5}, \\ \psi''' \sim \mp 480(\alpha \pm \theta)^{-6}, \qquad \psi^{(4)} \sim 2880(\alpha \pm \theta)^{-7}, \end{gathered} \tag{3.28}$$

as θ tends to $\mp\alpha$. With this information we conclude that h, h' and h'' are continuous in $(-\alpha+\varepsilon_0, \alpha-\varepsilon_0)$.
From (3.28) we obtain their behaviour near the boundary. So we see that in (3.27) the term $3ff'' - 4f'^2$ vanishes as θ tends to $\pm\alpha$. In the same way the highest singularities of h_1 and h_2 vanish, and we get

$$h^{(k)} \sim c_k(\alpha \pm \theta)^{3-k}, \qquad \theta \to \mp\alpha, \qquad k = 0, 1, 2$$

for and some constants $c_k \neq 0$, depending only on k. And so *a*) and *b*) are shown.
The property *c*) is an easy consequence of *a*) and *b*). What remains to show is point *d*).
Since the denominator of h' is positive, it is sufficient to show $f'h_1 > 0$, as θ tends to $\pm\alpha$. A close examination yields that we have, for a positive c:

$$\begin{aligned} h_1 &= c\psi^2(3\psi^2\psi''' - 8\psi\psi'\psi'' + 5\psi'^3) + O([\alpha \pm \theta]) \\ &= \mp c(\alpha \pm \theta)^{-18} + O([\alpha \pm \theta]). \end{aligned}$$

And so we have

$$f'h_1 = c(\alpha \pm \theta)^{-22} + O([\alpha \pm \theta]^{-3}), \qquad \text{as } \theta \to \mp\alpha,$$

and therefore

$$f'h' = c(\alpha \pm \theta)^{-2} + O([\alpha \pm \theta]), \qquad \text{as } \theta \to \mp\alpha,$$

which is positive as θ tends to $\pm\alpha$. A similar examination yields $fh = -c + O([\alpha \pm \theta])$ for a constant $c > 0$. So *d*) is shown.
For the sake of completeness we give the upper mentioned but omitted details. The mentioned

polynomials are

$$
\begin{aligned}
h_1 &= -\frac{3}{\kappa}(-27f^4f''' + 72f^3f'f'' - 3f^2f'[15f'^2 + f'f''' - 3f''^2] - 10ff'^3f'' + 4f'^5)\\
h_2 &= -\frac{3}{\kappa}(-243f^6f^{(4)} + 81f^5[11f'f''' + 8f''^2] - 27f^4[f'^2\{93f'' + 2f^{(4)}\} - 9f'f''f''' - 3f''^3]\\
&\quad + 9f^3[135f'^4 - 5f'^3f''' - 89f'^2f''^2] + 3f^2f'^2[f'^2\{282f'' - f^{(4)}\} + 9f'f''f''' - 12f''^3]\\
&\quad - 2ff'^4[135f'^2 + 8f'f''' - 19f''^2] - 10f'^6f'').
\end{aligned}
\tag{3.29}
$$

In (3.29) the terms

$$
\begin{aligned}
&-3f^2f'[f'f''' - 3f''^2] - 10ff'^3f'' + 4f'^5 \qquad \text{and}\\
&3f^2f'^2[-f'^2f^{(4)} + 9f'f''f''' - 12f''^3] - 2ff'^4[8f'f''' - 19f''^2] - 10f'^6f''
\end{aligned}
$$

vanish as θ tends to $\pm\alpha$.
To compute the third and fourth derivative of f we need:

$$
\begin{aligned}
\psi'''(\theta) &= \frac{3}{2}\sin^{-4}\left(\frac{\alpha-\theta}{2}\right) + \frac{15}{2}\cos^2\left(\frac{\alpha-\theta}{2}\right)\sin^{-6}\left(\frac{\alpha-\theta}{2}\right)\\
&\quad - \frac{3}{2}\sin^{-4}\left(\frac{\alpha+\theta}{2}\right) - \frac{15}{2}\cos^2\left(\frac{\alpha+\theta}{2}\right)\sin^{-6}\left(\frac{\alpha+\theta}{2}\right),\\
\psi^{(4)}(\theta) &= \frac{21}{2}\cos\left(\frac{\alpha-\theta}{2}\right)\sin^{-5}\left(\frac{\alpha-\theta}{2}\right) + \frac{45}{2}\cos^3\left(\frac{\alpha-\theta}{2}\right)\sin^{-7}\left(\frac{\alpha-\theta}{2}\right)\\
&\quad + \frac{21}{2}\cos\left(\frac{\alpha+\theta}{2}\right)\sin^{-5}\left(\frac{\alpha+\theta}{2}\right) + \frac{45}{2}\cos^3\left(\frac{\alpha+\theta}{2}\right)\sin^{-7}\left(\frac{\alpha+\theta}{2}\right),
\end{aligned}
$$

since we have $f = -\frac{\tilde{c}}{\kappa}\psi$ and $-\frac{\tilde{c}}{\kappa} > 0$. $\square$

Outlook

Of cause there is still a lot of work to do. First of all, the existence of a solution of problem (2.2) is not assured, neither for the general case nor for any special case. Maybe the work of Johnson and Perko [JP68] will provide a method to obtain an existence result for the circular case.

In this work we just accounted for some special cases. A still open problem is to expand our results on capillary tubes with arbitrary cross-sections.

An other problem is to find some solutions in the absence of gravity.

We computed the solution for the wedge problem just for the special case $q = 6$. Of course this is the most important case. But it is self-evident to compute the disjoining pressure potential for arbitrary $p > 3$ and apply the above developed method to achieve a general result. It is probably impossible to get an exact result for the disjoining pressure potential. It maybe also responsible, that the asymptotic behaviour of P is be sufficient to reach an asymptotic result for the solution of (2.2).

After determining an asymptotic solution, another goal is to find an asymptotic expansion for the problem (2.2) and the corresponding domain Ω. The Comparison Principle will be adjuvant thereby. Compare the articles of Miersemann [Mie94] and [Mie93a] for the case of a circular and a wedge domain respectively.

In reality, there is a variety of different pores in porous media. To apply our mathematical results in chemistry, it will not be sufficient to determine the shape of liquid layers on capillary tubes that is, on straight cylinders over some domain Ω. There are a lot of pores that can not be modeled by cylinders. For example spherical pores or slit cavities. A last goal is to develop a method to deal with other kind of pores.

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Leipzig, den 29. November 2009

....
(Oliver Thomys)

Wissenschaftlicher Werdegang

Februar 2010	Abschluss des Ergänzungsstudiums Astronomie Thema der Abschlussarbeit: "Polarlichter"
seit Juli 2007	Promotionsstudium an der Universität Leipzig Stipendiat im Graduiertenkolleg "Analysis, Geometrie und ihre Verbindung zu den Naturwissenschaften"
Mai 2007	Diplom in Mathematik Thema der Diplomarbeit: "Entropiezahlen für Diagonaloperatoren in Lorentzräumen"
April 2007	Studium an der Martin - Luther - Universität Halle - Wittenberg im Studiengang Astronomie
Oktober 2002	Studium an der Universität Leipzig im Diplomstudiengang Mathematik mit Nebenfach Physik

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